

MATH 310.01 - METHODS OF PROOF
EXAM II

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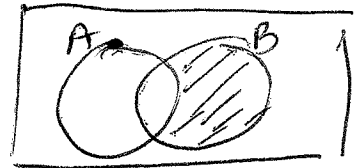
NAME: Key

SHOW ALL WORK NEATLY and clearly indicate how you arrived at your answers. If you aren't sure if you have written enough, please ask. There are 8 problems on 5 pages. The time allowed is 50 minutes and the exam is worth 100 points. Put your name on each page. If you need extra space, complete the problem on the back of the page and indicate that you have done so. GOOD LUCK!

1. (12 points) Suppose that A and B are subsets of a universal set U . Use the established set relationships to prove that

$$(A \cup B) - A = B - (A \cap B).$$

$$\begin{aligned}
B - (A \cap B) &= B \cap \overline{(A \cap B)} \\
&= B \cap (\overline{A} \cup \overline{B}) \\
&= (B \cap \overline{A}) \cup (B \cap \overline{B}) \\
&= (B \cap \overline{A}) \cup \emptyset \\
&= B \cap \overline{A} \\
&= (B \cap \overline{A}) \cup (A \cap \overline{A})
\end{aligned}$$



$$\begin{aligned}
&= (B \cup A) \cap \overline{A} \\
&= (A \cup B) - A
\end{aligned}$$

2. (a) (6 points) Express the following statement in symbols: "There exists an integer k such that for every nonnegative integer n , $k < n$."

$$\exists k \in \mathbb{Z}, \forall n \in \mathbb{N} \cup \{0\}, k < n.$$

- (b) (6 points) Prove that the statement in part (a) is true.

Let $k = -1$.



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3. Consider the statement "For an integer n , if $n^2 + 3n + 7$ is even then n is even."

(a) (4 points) Write the contrapositive of this statement.

If n is odd then $n^2 + 3n + 7$ is odd

(b) (8 points) Prove that this statement is true. (Hint: Consider part (a).) Use contrapositive.

Let n be an odd integer.

Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} \text{So } n^2 + 3n + 7 &= (2k + 1)^2 + 3(2k + 1) + 7 \\ &= 4k^2 + 4k + 1 + 6k + 3 + 7 \\ &= 2(2k^2 + 5k + 5) + 1 \end{aligned}$$

As $2k^2 + 5k + 5 \in \mathbb{Z}$, $n^2 + 3n + 7$ is odd.

By proving contrapositive, we have shown that
if $n^2 + 3n + 7$ is even, n is even.

4. (8 points) Disprove the following statement: "For every positive integer n , $3^n + 2^n$ is prime."

↓
Consider $n = 3$.

$$\text{Then } 3^3 + 2^3 = 27 + 8 = 35 = 7(5) \text{ is}$$

not prime.

So $n = 3$ is a counterexample.

5. Consider the statement: "For real numbers x and y , if x is a rational number and $5x + 2y$ is irrational then y is an irrational number.

(a) (5 points) To prove this result by contradiction, what would you assume?

Let $x, y \in \mathbb{R}$ & x is rational, $5x + 2y$ is irrational $\therefore y$ is rational.

(b) (12 points) Prove the statement.

Assume the above.

Then $\exists j, k, m, n \in \mathbb{Z}$ & $k, n \neq 0$, $x = \frac{j}{k}$ & $y = \frac{m}{n}$.

$$\text{So } 5x + 2y = 5\left(\frac{j}{k}\right) + 2\left(\frac{m}{n}\right)$$

$$= \frac{5jn + 2mk}{kn}$$

As $5jn + 2mk$, $kn \in \mathbb{Z}$ and $kn \neq 0$,

$5x + 2y$ is rational.

But $5x + 2y$ is irrational, so this is a contradiction.

Thus if x is rational and $5x + 2y$ is irrational, y is irrational.

6. (12 points) Let A and B and \bar{A} be subsets of some universal set U . Prove that

$$(A \cap \bar{B}) \subseteq \overline{A \cap B}.$$

Let $x \in A \cap \bar{B}$

Then $x \in A$ and $x \in \bar{B}$

So $x \in A$ and $x \notin B$

So $x \notin A \cap B$

Then $x \in \overline{A \cap B}$

Thus $(A \cap \bar{B}) \subseteq \overline{A \cap B}$.

7. (10 points) Let a, b be integers such that $a \equiv 5 \pmod{6}$ and $b \equiv 2 \pmod{3}$. Prove that $ab \equiv 1 \pmod{3}$.

As $a \equiv 5 \pmod{6}$, $6 \mid (a-5)$, so $a-5 = 6k$ for some $k \in \mathbb{Z}$.

As $b \equiv 2 \pmod{3}$, $3 \mid (b-2)$, so $b-2 = 3j$ for some $j \in \mathbb{Z}$.

Thus $a = 5 + 6k$ and $b = 2 + 3j$.

$$\text{So } ab = (5+6k)(2+3j)$$

$$= 10 + 12k + 15j + 18kj$$

$$= 9 + 12k + 15j + 18kj + 1$$

$$= 3(3 + 4k + 5j + 6kj) + 1$$

$\therefore ab - 1 = 3(3 + 4k + 5j + 6kj)$. As $3 + 4k + 5j + 6kj \in \mathbb{Z}$,

we have $3 \mid (ab - 1)$

Thus $ab \equiv 1 \pmod{3}$.

8. For the sequence $a_0, a_1, a_2, \dots, a_n, \dots$, assume that $a_1 = 5$, and for each natural number $n \geq 2$, $a_n = 3a_{n-1} + 2$.

(a) (4 points) Give the first 4 terms of the sequence.

$$a_1 = 5$$

$$a_2 = 3(5) + 2 = 17$$

$$a_3 = 3(17) + 2 = 53$$

$$a_4 = 3(53) + 2 = 161$$

(b) (15 points) Use mathematical induction to prove that, for each positive integer n , $a_n = 6(3^{n-1}) - 1$.

Basis case ($n=1$). ✓

Notice $a_1 = 5$ and $6(3^{1-1}) - 1 = 6(1) - 1 = 5$, ✓✓

So the formula holds for $n=1$.

Inductive hypothesis

Suppose $a_k = 6(3^{k-1}) - 1$ for some $k \geq 1$. ✓

Inductive step.

Consider $n=k+1$. ✓

$$a_{k+1} = 3a_k + 2 \quad \checkmark \checkmark$$

$$= 3(6(3^{k-1}) - 1) + 2 \quad \text{by the inductive hypothesis}$$

$$= 6(3^k) - 3 + 2$$

$$= 6(3^k) - 1 \quad \checkmark \checkmark$$

So the formula holds for $n=k+1$.

Thus by the principle of mathematical

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induction, $a_n = 6(3^{n-1}) - 1 \quad \forall n \in \mathbb{N}$. ✓