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MATH 310.01 - METHODS OF PROOF
EXAM III

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SHOW ALL WORK NEATLY and clearly indicate how you arrived at your answers. If you aren't sure if you have written enough, please ask. There are 9 problems on 5 pages. The time allowed is 50 minutes and the exam is worth 100 points. *Put your name on each page.* If you need extra space, complete the problem on the back of the page and indicate that you have done so. GOOD LUCK!

1. Let $A = \{a, b, c, d\}$ and let f be the function on A defined by

$$f = \{(a, c), (b, d), (c, c), (d, a)\}.$$

(a) (4 points) Determine f^{-1}.

$f^{-1} = \{(c,a), (d,b), (c,c), (a,d)\}$

(b) (4 points) Is f^{-1} a function? Why or why not?

No. $f^{-1}(c) = a$ and $f^{-1}(c) = c$.

2. (8 points) In $\mathbf{Z}_7$, express the following as $[r]$, where $0 \le r < 7$.

(a) $[-43] + [101]$

$= [6] + [3]$

$= [9]$

$= [2]$

(b) $[25] \cdot [-37]$

$= [4] \cdot [5]$

$= [20]$

$= [6]$

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3. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Define the relation R on S as follows:

For all $a, b \in S, aRb$ if and only if $5a \equiv 2b \pmod 3$.

(a) (15 points) Prove that R is an equivalence relation.

Let $a \in S$. Then $5a - 2a = 3a$, so $5a \equiv 2a \pmod 3$.
Thus aRa & R is reflexive.
Let $a, b \in S$ such that aRb. Then $5a \equiv 2b \pmod 3$,
so $3 \mid (5a-2b)$. Thus $\exists k \in \mathbb{Z}$ st $5a - 2b = 3k$.
Now $5b - 2a = 3b + 3a - (5a-2b) = 3b + 3a - 3k = 3(b+a-k)$.
As $b+a-k \in \mathbb{Z}$, $3 \mid 5b-2a$, so $5b \equiv 2a \pmod 3$.
Thus bRa and R is symmetric.
Let $a, b, c \in S$ such that aRb and bRc. Then
$5a \equiv 2b \pmod 3$ and $5b \equiv 2c \pmod 3$. So $3 \mid (5a-2b)$
and $3 \mid (5b-2c)$. Thus $\exists k, j \in \mathbb{Z}$ st $5a - 2b = 3k$
and $5b - 2c = 3j$. Now
$5a - 2c = (5a-2b) + (5b-2c) - 3b = 3k + 3j - 3b = 3(k+j-b)$.
As $k+j-b \in \mathbb{Z}$, $3 \mid (5a-2c)$, so $5a \equiv 2c \pmod 3$. Thus
aRc and R is transitive.
As R is reflexive, symmetric & transitive, R is an equivalence relation

(b) (6 points) Determine all the distinct equivalence classes of this equivalence relation.

$\{0, 3, 6\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$

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4. Define $f : \mathbf{Z}_{\cancel{8}\,6} \to \mathbf{Z}_{\cancel{8}\,6}$ by $f([a]) = [5a+1]$.

(a) (8 points) Show that f is well-defined.

Let $[a], [b] \in \mathbb{Z}_{6}$ st $[a]=[b]$.

Then $a \equiv b \pmod 6$, so $5a \equiv 5b \pmod 6$.

As $1 \equiv 1 \pmod 6$, $5a+1 \equiv (5b+1) \pmod 6$,

and so $[5a+1] = [5b+1]$.

Thus $f([a]) = f([b])$ and f is well-defined.

(b) (6 points) Is f ~~a surjection~~ an injection? Why or why not?

$f([0]) = [1]$

$f([1]) = [0]$

$f([2]) = [5]$

$f([3]) = [4]$

$f([4]) = [3]$

$f([5]) = [2]$

Yes, there are 6 distinct images under f.

5. (8 points) Let A, B, and C be nonempty sets and let $f : A \to B$ and $g : B \to C$. Suppose that $g \circ f$ is an injection. Prove that f is an injection.

Let $x, y \in A$ and suppose $f(x) = f(y)$

Then, as g is a function,

$$(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y).$$

As $g \circ f$ is an injection, $x = y$.

Thus f is one-to-one

ie: f is an injection.

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6. (15 points) Define $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $g(x, y) = (y - 7, 2x + 3)$ Prove that g is a bijection.

Let $(x,y),(z,w) \in \mathbb{R}\times\mathbb{R}$ and suppose $g(x,y) = g(z,w)$.
Thus $(y-7, 2x+3) = (w-7, 2z+3)$ and so we have
$y-7 = w-7$ and $2x+3 = 2z+3$. Thus $y = w$ and $x = z$
ie: $(x,y) = (z,w)$, and we have shown that g is
one-to-one.

Let $(z,w) \in \mathbb{R}\times\mathbb{R}$. We need $(x,y) \in \mathbb{R}\times\mathbb{R}$ st $g(x,y) = (z,w)$.
Consider $(x,y) = \left(\frac{w-3}{2}, z+7\right)$. Then $(x,y) \in \mathbb{R}\times\mathbb{R}$
and $g(x,y) = g\left(\frac{w-3}{2}, z+7\right) = \left((z+7)-7, 2\left(\frac{w-3}{2}\right)+3\right) = (z,w)$.
Thus g is onto.
As g is one-to-one and onto, g is a bijection.

7. (8 points) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. The distinct equivalence classes resulting from an equivalence relation R are $\{1, 4, 6\}$, $\{3, 5\}$, $\{2\}$ and $\{7\}$. What is R?

$R = \{(1,1), (1,4), (4,1), (4,4), (1,6), (6,1), (4,6),$
$(6,4), (6,6), (3,3), (3,5), (5,3), (5,5),$
$(2,2), (7,7)\}$

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8. (12 points) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 5 & 1 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix}$ be permutations in S_6.

(a) Determine $\alpha \circ \beta$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

(b) Determine β^{-1}.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 4 & 2 & 6 \end{pmatrix}$$

(c) Give another permutation in ~~S_5~~ S_6 aside from the two given above.

9. (8 points) Let $f : \mathbb{R} - \{3\} \to \mathbb{R}$ be defined by $f(x) = \frac{4}{x-3}$. Is f a surjection? Why or why not?

Consider $y \in \mathbb{R}$.

Then $\frac{4}{x-3} = y$ if $4 = xy - 3y$, and so if $x = \frac{4+3y}{y}$.

Thus we cannot have $y = 0$

(Also notice, $\frac{4}{x-3} \neq 0 \; \forall x$ as $4 \neq 0$.)

$\therefore$ f is <u>not</u> a surjection.

THE END

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