MATH 310.01 - METHODS OF PROOF
HOMEWORK 12 KEY

Chapter 9: 28b; Chapter 10: 4
Due: December 11, 2009

Question 9.28b

For nonempty sets A, B, and C, let $f : A \to B$ and $g : B \to C$ be functions. Disprove: If $g \circ f$ is one-to-one, then g is one-to-one:
Consider $A = (0,1)$, $B = C = \mathbb{R}$, $f(x) = x$ and $g(x) = x^2$. Then $(g \circ f) : (0,1) \to \mathbb{R}$ is given by $(g \circ f)(x) = x^2$, which is one-to-one on $(0,1)$. However, $g : \mathbb{R} \to \mathbb{R}$ is not one-to-one as $g(1) = g(-1) = 1$. (Note: this is only one possible example. There are many others.)

Question 10.4

Prove that the function $f : \mathbb{N} \to \mathbf{Z}$ defined by $f(n) = \dfrac{1 + (-1)^n(2n-1)}{4}$ is bijective.
We will prove that f is one-to-one and onto.
We will first show that f is one-to-one. Let $m, n \in \mathbb{N}$ and suppose that $f(n) = f(m)$. Thus

$$\frac{1 + (-1)^n(2n-1)}{4} = \frac{1 + (-1)^m(2m-1)}{4}.$$

Multiplying through by 4 and subtracting 1 gives $(-1)^n(2n-1) = (-1)^m(2m-1)$. We now consider cases on m and n.

- Case 1: Suppose m and n are of the same parity. Then $(-1)^n = (-1)^m$, and so we have $2n - 1 = 2m - 1$. Adding 1 and dividing through by 2 gives $n = m$.
- Case 2: Suppose m and n are of opposite parity. Without loss of generality, suppose m is odd and n is even. Then we have $2n - 1 = 1 - 2m$, or $n = 1 - m$. However, $n, m \in \mathbb{N}$, and there are no natural numbers n, m such that $n = 1 - m$. Thus it is vacuously true that $n = m$.

In either case, we showed that $n = m$. Thus we have that f is one-to-one.
We now show that f is onto. Let $z \in \mathbf{Z}$. We need to find $n \in \mathbb{N}$ such that $f(n) = z$. That is, we need $\dfrac{1 + (-1)^n(2n-1)}{4} = z$. Multiplying by 4 and subtracting 1, we see that we need n such that $(-1)^n(2n-1) = 4z-1$.
Now consider cases on z.

- Case 1: Suppose $z > 0$. Consider $n = 2z$. As $z > 0$, $n = 2z \in \mathbb{N}$. Also,
$$f(n) = \frac{1 + (-1)^n(2n-1)}{4} = \frac{1 + (-1)^{2z}(2(2z)-1)}{4} = \frac{1 + (1)(4z-1)}{4} = z.$$
- Case 2: Suppose $z = 0$. Notice that $f(1) = 0$, and $1 \in \mathbb{N}$.
- Case 3: Suppose $z < 0$. Consider $n = 1 - 2z$. As $z < 0$, $n = 1 - 2z \in \mathbb{N}$. Also,
$$f(n) = \frac{1 + (-1)^n(2n-1)}{4} = \frac{1 + (-1)^{1-2z}(2(1-2z)-1)}{4} = \frac{1 + (-1)(1-4z)}{4} = z.$$

In each case, we found $n \in \mathbb{N}$ such that $f(n) = z$. Thus f is onto.
As f is one-to-one and onto, f is a bijection.