Appendix B

SETS AND FUNCTIONS

B.1 SETS AND THE ALGEBRA OF SETS

The concept of *set* is the most basic of all mathematical concepts. Indeed, it even precedes counting. In order to be able to count "how many," one must be able to conceive of the objects being counted as somehow separated from all objects *not* being counted. Thus, it is natural that the algebra of sets is logically placed before the algebra of numbers. This section will review the basic concepts, operations, and relations of sets.

We first point out that the word "set" is an undefined term in our context. That is, we assume that the reader has an understanding of the word that conforms to certain axioms, and does not require further definition. To describe these axioms is beyond the scope of this book. One of them, however, states that a set is completely determined by its *elements* (or its *members*); that is, by what *belongs to* it.

NOTATION: We usually denote sets by capital letters and their elements by lower case letters.

The symbol "$\in$" is used to denote "is an element of". Thus,

$$x \in A$$

is the statement "x is an element of A," or "x belongs to A."

B.1.1 Definition. *If $p(x)$ is any proposition about a variable x, then $\{x : p(x)\}$ denotes the set of all values of x for which $p(x)$ is true. It is sometimes called the "truth set" of $p(x)$.*

B.1.2 Definition. *n any particular context in which variables are used, there is a* **universal set,** *$\mathcal{U}$, from which the variables draw their values. This $\mathcal{U}$ is often understood without explicit mention. For example, when you see an equation like $3x^2+7x-10=0$, you assume that $\mathcal{U}$ is a set of numbers, usually either the set of all real numbers or the set of all complex numbers. To* **solve** *an equation* *is to find all values in $\mathcal{U}$ that, when substituted for the unknown(s), make the equation true.*

The **empty set**, *$\emptyset$, is the set that has no members. Thus, for example, $\emptyset = \{x : x \neq x\}$.*

B.1.3 Definition. *Some special sets.* *Although the official definitions of natural numbers, integers, rational numbers, and so on are not given until Chapter 1, we shall use the following symbols for the sets of these familiar types of numbers:*

$$\begin{aligned}
&\mathbb{N} = \{\text{all natural numbers}\} = \{1, 2, 3, 4, \cdots\};\\
&\mathbb{Z} = \{\text{all integers}\} = \{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\};\\
&\mathbb{Q} = \{\text{all rational numbers}\} = \left\{\tfrac{m}{n} : m, n \in \mathbb{Z}, \text{ and } n \neq 0\right\};\\
&\mathbb{R} = \{\text{all real numbers}\} = \{\text{all numbers located on a ``number line''}\}.
\end{aligned}$$

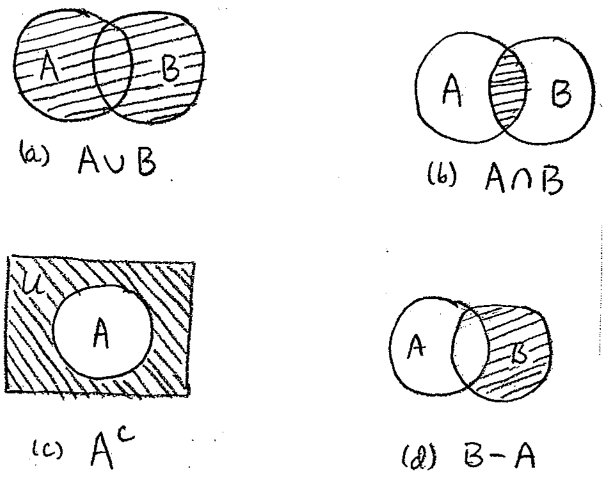
We also use the interval notation familiar from calculus: $\forall a, b \in \mathbb{R}$,

$$\begin{array}{ll}
(a,b) = \{x \in \mathbb{R} : a < x < b\}; & (-\infty, b) = \{x \in \mathbb{R} : x < b\};\\
[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}; & (-\infty, b] = \{x \in \mathbb{R} : x \leq b\};\\
(a,b] = \{x \in \mathbb{R} : a < x \leq b\}; & (a, +\infty) = \{x \in \mathbb{R} : x > a\};\\
[a,b) = \{x \in \mathbb{R} : a \leq x < b\}; & [a, +\infty) = \{x \in \mathbb{R} : x \geq a\};\\
 & (-\infty, \infty) = \mathbb{R}.
\end{array}$$

B.1.4 Definition. *Let A and B be sets. Then*

(a) The **union** *of A and B is the set $\mathbf{A} \cup \mathbf{B} = \{x : x \in A \text{ or } x \in B\}$.*
(b) The **intersection** *of A and B is the set $\mathbf{A} \cap \mathbf{B} = \{x : x \in A \text{ and } x \in B\}$.*
(c) The **complement** *of A is the set $\mathbf{A}^c = \{x \in \mathcal{U} : x \notin A\}$.*
(d) The **relative complement** *of A in B is the set $\mathbf{B} - \mathbf{A} = \{x \in B : x \notin A\}$.*

These sets are conveniently illustrated in the following "Venn diagrams:"



(a) $A \cup B$

(b) $A \cap B$

(c) A^c

(d) $B - A$

Figure 1

B.1.5 Example. *Let $\mathcal{U} = \{$real numbers$\}$. Then*

$$\begin{aligned}
[0,3] \cup [1,5] &= [0,5] \\
[0,3] \cap [1,5] &= [1,3] \\
[0,3]^c &= (-\infty, 0) \cup (3, +\infty) \\
[0,3] - [1,5] &= [0,1) \\
[1,5] - [0,3] &= (3,5]
\end{aligned}$$

B.1.6 Definition. *We say that* $\mathbf{A \subseteq B}$ *(A is a* **subset** *of B) iff every element of A is also an element of B.*

For example, $\{1,2,3\} \subseteq \mathbb{N}$ and $\{x : x^2 - 3x + 2 = 0\} \subseteq \mathbb{N}$, but $[1,3] \nsubseteq \mathbb{N}$.

B.1.7 Theorem. ***(Algebra of sets)****: For any sets $A, B, C \in \mathcal{U}$,*
(a) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.
(b) $\emptyset \subseteq A$, $A \subseteq A$, and $A \subseteq \mathcal{U}$.
(c) $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
(d) $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
(e) $A \cup B = A$ iff $B \subseteq A$.
(f) $A \cap B = A$ iff $A \subseteq B$.
(g) $(A \cup B)^c = A^c \cap B^c$. (de Morgan's law)
(h) $(A \cap B)^c = A^c \cup B^c$. (de Morgan's law)
(i) $A^c = \mathcal{U} - A$.
(j) $\mathcal{U}^c = \emptyset$; and $\emptyset^c = \mathcal{U}$
(k) $B - A = B \cap A^c$.
(l) $A^{cc} = A$.
(m) $A \cup (B \cup C) = (A \cup B) \cup C$. (associative law for $\cup$)
(n) $A \cap (B \cap C) = (A \cap B) \cap C$. (associative law for $\cap$)
(o) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (distributive law)
(p) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (distributive law)

Proof. of (g):

Part 1: Suppose $x \in (A \cup B)^c$. Then $x \in \mathcal{U}$ but $x \notin A \cup B$. Then it is not true that $x \in A$ or $x \in B$. By de Morgan's rule in logic, this means $x \notin A$ and $x \notin B$. That is, $x \in A^c$ and $x \in B^c$; i.e., $x \in A^c \cap B^c$. Therefore, $(A \cup B)^c \subseteq A^c \cap B^c$.

Part 2: Suppose $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$; i.e., $x \notin A$ and $x \notin B$. By de Morgan's rule in logic, this means it is not true that $x \in A$ or $x \in B$. Then $x \in \mathcal{U}$ but $x \notin A \cup B$; i.e., $x \in (A \cup B)^c$. Therefore, $A^c \cap B^c \subseteq (A \cup B)^c$.

By parts 1 and 2, together with Theorem B.1.1 (a), $(A \cup B)^c = A^c \cap B^c$.

Proof of (o):

Part 1: Suppose $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Then $x \in A$ and ($x \in B$ or $x \in C$). By the distributive law in logic [see Theorem A.1.23 (a)] this means ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). That is, $x \in A \cap B$ or $x \in A \cap C$. Thus, $x \in (A \cap B) \cup (A \cap C)$.

Part 2: Exercise 5. ■

While the above theorem summarizes the algebra of sets when only several sets are involved, we need algebraic rules to cover situations in which many, even infinitely many, sets are involved. The following definition and theorem cover these situations.

B.1.8 Definition. ***(Operations on collections of sets):*** *Let $\mathcal{C} = \{A_\lambda : \lambda \in \Lambda\}$ be a collection of sets, "indexed" by some set Λ of "indices." Then*

(a) $\cap\mathcal{C} = \bigcap_{\lambda\in\Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for every } \lambda \in \Lambda\}$.

(b) $\cup\mathcal{C} = \bigcup_{\lambda\in\Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for at least one } \lambda \in \Lambda\}$.

B.1.9 Example. *(a)* $\cap\{(-\frac{1}{n}, 1+\frac{1}{n}) : n \in \mathbb{N}\} = \bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, 1+\frac{1}{n}) = [0,1]$.

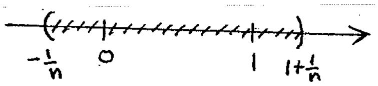


Figure 2

(b) $\cup\{(-\frac{1}{n}, 1+\frac{1}{n}) : n \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}} (-\frac{1}{n}, 1+\frac{1}{n}) = (-1,2)$.

B.1.10 Theorem. ***(Algebra of collections of sets):*** *Let $\mathcal{C} = \{A_\lambda : \lambda \in \Lambda\}$ be a collection of sets and let B be any set. Then*

(a) $\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$. *(de Morgan's law)*

(b) $\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} A_\lambda^c$. *(de Morgan's law)*

(c) $B \cup \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} (B \cup A_\lambda)$. *(distributive law)*

(d) $B \cap \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} (B \cap A_\lambda)$. *(distributive law)*

(e) $B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} (B - A_\lambda)$. *(de Morgan's law)*

(f) $B - \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} (B - A_\lambda)$. *(de Morgan's law)*

Proof. of (e): Let $\mathcal{C} = \{A_\lambda : \lambda \in \Lambda\}$ be a collection of sets and let B be any set.

Part 1: Suppose $x \in B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. Then $x \in B$ but $x \notin \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. That is, $x \in B$ but $\sim \forall\lambda \in \Lambda, x \in A_\lambda$. By "quantifier negation[1]". this means $x \in B$ but $\exists\,\lambda \in \Lambda \ni x \notin A_\lambda$. That is, $x \in B$ but $\exists\,\lambda \in \Lambda \ni x \in A^c_\lambda$. Equivalently, $\exists\,\lambda \in \Lambda \ni x \in B \cap A^c_\lambda$. Equivalently, $\exists\,\lambda \in \Lambda \ni x \in B - A_\lambda$. But that means $x \in \bigcup_{\lambda\in\Lambda}(B - A_\lambda)$. Therefore, $B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \subseteq \bigcup_{\lambda\in\Lambda}(B - A_\lambda)$.

Part 2: Suppose $x \in \bigcup_{\lambda\in\Lambda}(B - A_\lambda)$. Then $\exists\,\lambda \in \Lambda \ni x \in B - A_\lambda$. That is, $\exists\,\lambda \in \Lambda \ni x \in B \cap A^c_\lambda$. Equivalently, $x \in B$ but $\exists\,\lambda \in \Lambda \ni x \in A^c_\lambda$. By quantifier negation, this means $x \in B$ but $\sim \forall\lambda \in \Lambda, x \in A_\lambda$. That is, $x \in B$ but $x \notin \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$; i.e., $x \in B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. Therefore, $\bigcup_{\lambda\in\Lambda}(B - A_\lambda) \subseteq \bigcup_{\lambda\in\Lambda}(B - A_\lambda)$.

By parts 1 and 2, together with Theorem B.1.1 (a), $B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda}(B - A_\lambda)$. ■

EXERCISE SET B.1

1. In each of the following, a universal set $\mathcal{U}$ and sets A, B, and C are given. Find $A \cap B$, $A \cup B$, A^c, B^c, $A - B$, $B - A$, $A \cup (B \cap C)$, and $A \cap (B \cup C)$.
   (a) $\mathcal{U} = \{1, 2, 3, \cdots, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, and $C = \{3, 4, 5\}$.
   (b) $\mathcal{U} = \{1, 2, 3, \cdots, 10\}$, $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{2, 4, 6, 8, 10\}$.
   (c) $\mathcal{U} = \{\text{all real numbers}\}$, $A = (0, 4)$, $B = [3, 6]$, $C = (2, 5)$.
   (d) $\mathcal{U} = \{\text{all real numbers}\}$, $A = (-\infty, 2)$, $B = [1, +\infty)$, $C = (-1, 1)$.
2. Prove Theorem B.1.7 (e).
3. Prove Theorem B.1.7 (h).
4. Prove Theorem B.1.7 (k).
5. Finish the proof of Theorem B.1.7 (o) by proving "Part 2."
6. Prove Theorem B.1.7 (p).
7. In each of the following, a collection of sets $\{A_\lambda : \lambda \in \Lambda\}$ is given. Assume $\mathcal{U} = \{\text{all real numbers}\}$. Find $\bigcap_{\lambda\in\Lambda} A_\lambda$, $\bigcup_{\lambda\in\Lambda} A_\lambda$, $\bigcup_{\lambda\in\Lambda} A^c_\lambda$, and $\bigcup_{\lambda\in\Lambda} A^c_\lambda$. In each case, verify Theorem B.1.10 (a) or (b).
   (a) $\{A_\lambda : \lambda \in \Lambda\} = \{(-n, n) : n \in \mathbb{N}\}$.

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[1] See section A.2.

(b) $\{A_\lambda : \lambda \in \Lambda\} = \{(-\infty, n) : n \in \mathbb{N}\}$.

(c) $\{A_\lambda : \lambda \in \Lambda\} = \{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$.

(d) $\{A_\lambda : \lambda \in \Lambda\} = \{[-2+\frac{1}{n}, 2-\frac{1}{n}] : n \in \mathbb{N}\}$.

(e) $\{A_\lambda : \lambda \in \Lambda\} = \{(n, n+1) : n \in \mathbb{N}\}$.

8. Prove Theorem B.1.10 (a).
9. Prove Theorem B.1.10 (b).
10. Prove Theorem B.1.10 (c).
11. Prove Theorem B.1.10 (d).
12. Prove Theorem B.1.10 (f).

B.2 FUNCTIONS

BASIC CONCEPTS OF FUNCTIONS

B.2.1 Definition. *If A and B are sets, a* **function** *f from A to B is any rule of correspondence which associates to each element $a \in A$ a unique element $f(a) \in B$. The set A is called the* **domain** *of f, and the set B is called the* **codomain** *of f. The set $\mathcal{R}(f) = \{f(a) : a \in A\}$ is called the* **range** *of f. We often denote the domain of f by $\mathcal{D}(f)$. The range of a function is a subset of its codomain.*

The notational phrase

$$f : A \to B$$

is often used as a sentence saying that "f is a function from set A to set B". It is also used as a noun, referring to "the function f from A to B." Context will determine which of the two uses is intended.

A function $f : A \to B$ may be viewed intuitively as an input/output relation. To each input $a \in A$ there corresponds a unique output $f(a) \in B$. The set of all inputs is A, or $\mathcal{D}(f)$, and the set of all outputs is $\mathcal{R}(f)$.

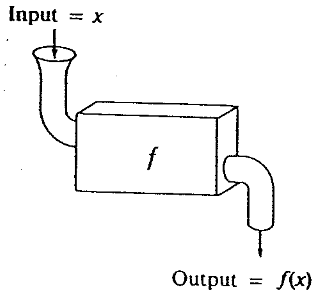

Figure 3

B.2.2 Definition. *Two functions* $f : A \to B$ *and* $g : A' \to B'$ *are said to be* **equal** *if* $A = A'$, $B = B'$, *and* $\forall x \in A$, $f(x) = g(x)$.

B.2.3 Definition. *A function* $f : A \to B$ *is* **one-to-one** *(or* **1-1***) if* $\forall a, a' \in A$, $f(a) = f(a') \Rightarrow a = a'$. *Equivalently,* $a \neq a' \Rightarrow f(a) \neq f(a')$.

Thus, a function $f : A \to B$ is 1-1 iff [2] no two inputs produce the same output.

B.2.4 Definition. *A function* $f : A \to B$ *is* **onto** B *if* $\mathcal{R}(f) = B$.

That is, $f : A \to B$ is onto B iff every element of B is an output of f.

B.2.5 Example. *The function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^2$ *is*

- *not 1-1, since* $f(1) = 1$ *and* $f(-1) = 1$;
- *not onto* $\mathbb{R}$, *since* $\mathcal{R}(f) = [0, +\infty) \neq \mathbb{R}$.

B.2.6 Example. *The function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^3$ *is*

- *1-1, since*

$$\begin{aligned} f(a) = f(b) &\Rightarrow a^3 = b^3 \\ &\Rightarrow a^3 - b^3 = 0 \\ &\Rightarrow (a-b)(a^2 + ab + b^2) = 0 \\ &\Rightarrow a - b = 0, \text{ since } a^2 + ab + b^2 \neq 0 \text{ (if } b \neq 0) \\ &\Rightarrow a = b. \end{aligned}$$

  Note that the reason $a^2 + ab + b^2 \neq 0$ *when* $b \neq 0$ *lies in the fact that* a *and* b *are real numbers, and the discriminant of the quadratic function* $g(a) = a^2 + ab + b^2$ *is* $D = b^2 - 4(1)(b^2) = -3b^2 < 0$.

- *onto* $\mathbb{R}$, *since* $\forall x \in \mathbb{R}$, $\exists \sqrt[3]{x} \in \mathbb{R}$ *and* $f(\sqrt[3]{x}) = x$, *so* $x \in \mathcal{R}(f)$.

B.2.7 Definition. *A function* $f : A \to B$ *that is both 1-1 and onto is said to be a* **1-1 correspondence.**

B.2.8 Definition. *Two sets* A *and* B *are said to have the* **same cardinal number** *(of elements) if* $\exists$ *1-1 correspondence* $f : A \to B$.

IMAGES AND INVERSE IMAGES OF SETS

B.2.9 Definition. *Suppose* $f : A \to B$ *is a function and* $C \subseteq A$ *and* $D \subseteq B$. *Then*

$$f(C) = \{f(x) : x \in C\};$$
$$f^{-1}(D) = \{x : f(x) \in D\}.$$

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[2] For the definition of "iff" see Definition A.1.9.

The set $f(C)$ is called the **image of** C under f and the set $f^{-1}(D)$ is called the **inverse image of** D under f. When we write $f^{-1}(D)$ we must be careful not to assume that f^{-1} is a function. Sometimes f^{-1} is a function, but that is a separate issue, to be discussed later.

Notice that $\mathcal{D}(f) = f^{-1}(B)$ and $\mathcal{R}(f) = f(A)$.

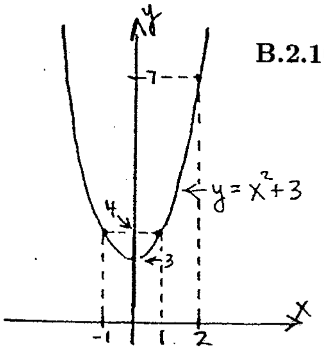

Figure 4

B.2.10 Example. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^2 + 3$. *Let* $C = [-1, 2]$ *and* $D = [0, 7]$. *Then*

$$f(C) = [3,7] \text{ and } f^{-1}(D) = [-2,2].$$

Observe that

$$f([-1,2]) = f([0,2]) = f([-2,2]) = [3,7],$$

and that

$$f^{-1}([0,7]) = f^{-1}([3,7]) = f^{-1}([-11,7]) = [-2,2].$$

Further, $f(\{2\}) = f(\{-2,2\} = \{7\}$, while $f^{-1}(\{7\}) = f^{-1}(\{-3,0,7\} = \{-2,2\}$.
■

B.2.11 Theorem. ***(Functions and sets):*** *Suppose* $f : A \to B$ *is a function. Then*

(a) $\forall C_1, C_2 \subseteq A$, $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

(b) $\forall C_1, C_2 \subseteq A$, $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$. *It is possible that* $f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$. *(See B.2.12 (b) below.)*

(c) $\forall C_1, C_2 \subseteq A$, $f(C_1) - f(C_2) \subseteq f(C_1 - C_2)$. *It is possible that* $f(C_1) - f(C_2) \neq f(C_1 - C_2)$. *(See B.2.12 (c) below.)*

(d) $\forall D_1, D_2 \subseteq B$, $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

(e) $\forall D_1, D_2 \subseteq B$, $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

(f) $\forall D_1, D_2 \subseteq B$, $f^{-1}(D_1 - D_2) = f^{-1}(D_1) - f^{-1}(D_2)$.

Proof. of (a): Suppose $f : A \to B$ is a function, and $C_1, C_2 \subseteq A$.

Part 1: Let $y \in f(C_1 \cup C_2)$. By definition, this means $\exists x \in C_1 \cup C_2 \ni f(x) = y$. But then, $\exists x \in C_1 \ni f(x) = y$, or $\exists x \in C_2 \ni f(x) = y$. That is, $y \in f(C_1)$ or $y \in f(C_2)$. Thus, $y \in f(C_1) \cup f(C_2)$. Therefore, $f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2)$.

Part 2: Let $y \in f(C_1) \cup f(C_2)$. Then $y \in f(C_1)$ or $y \in f(C_2)$. That is, $\exists x \in C_1 \ni f(x) = y$, or $\exists x \in C_2 \ni f(x) = y$. In both of these two cases, we can say $\exists x \in C_1 \cup C_2 \ni f(x) = y$. Thus, $y \in f(C_1 \cup C_2)$. Therefore, $f(C_1) \cup f(C_2) \subseteq f(C_1 \cup C_2)$.

By Parts 1 and 2, together with Theorem B.1.7 (a), $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

Proof of (e) Suppose $f : A \to B$ is a function, and $D_1, D_2 \subseteq B$.

Part 1: Let $x \in f^{-1}(D_1 \cap D_2)$. By definition of f^{-1}, this means $f(x) \in D_1 \cap D_2$. But then, $f(x) \in D_1$ and $f(x) \in D_2$. That is, $x \in f^{-1}(D_1)$ and $x \in f^{-1}(D_2)$. Thus, $x \in f^{-1}(D_1) \cap f^{-1}(D_2)$. Therefore, $f^{-1}(D_1 \cap D_2) \subseteq f^{-1}(D_1) \cap f^{-1}(D_2)$.

Part 2: Let $x \in f^{-1}(D_1) \cap f^{-1}(D_2)$. Then $x \in f^{-1}(D_1)$ and $x \in f^{-1}(D_2)$. That is, $f(x) \in D_1$ and $f(x) \in D_2$. But that means $f(x) \in D_1 \cap D_2$; i.e., $x \in f^{-1}(D_1 \cap D_2)$.

By Parts 1 and 2, together with Theorem B.1.7 (a), $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$. ∎

B.2.12 Example. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2 + 3$. *Then*

(a) $f((-1,0) \cup (0,2)) = f(-1,2) = (3,7)$, *and*
$f(-1,0) \cup f(0,2) = (3,4) \cup (3,7) = (3,7)$.

(b) $f((-1,0) \cap (0,2)) = f(\emptyset) = \emptyset$, *while*
$f(-1,0) \cap f(0,2) = (3,4) \cap (3,7) = (3,4)$.
Note: in this case, $f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$.

(c) $f((-1,1) - f(0,1)) = f(-1,0) = (3,4)$, *while*
$f(-1,1) - f(0,1) = (3,4) \cup (3,4) = \emptyset$.
Note: in this case, $f(C_1 - C_2) \neq f(C_1) - f(C_2)$.

(d) $f^{-1}((-\infty,3] \cup (2,4)) = f^{-1}(-\infty,4) = (-1,1)$, *and*
$f^{-1}(-\infty,3] \cup f^{-1}(2,4) = \{0\} \cup (-1,1) = (-1,1)$.

(e) $f^{-1}((-\infty,3] \cap (2,4)) = f^{-1}(2,3) = \{0\}$, *and*
$f^{-1}(-\infty,3] \cap f^{-1}(2,4) = \{0\} \cap (-1,1) = \{0\}$.

(f) $f^{-1}((-\infty,3] - (2,4)) = f^{-1}((-\infty,2] - [4,7]) = [-2,-1] \cup [1,2]$, *and*
$f^{-1}(-\infty,3] - f^{-1}(2,4) = [-2,2] - (-1,1) = [-2,-1] \cup [1,2]$.

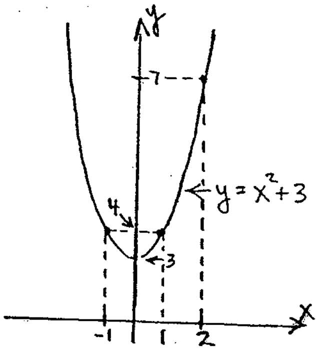

Figure 5

The following theorem generalizes Theorem B.2.11 to families of sets, even infinitely many sets.

B.2.13 Theorem. ***(Functions and collections of sets):*** *Suppose* $f : A \to B$ *is a function. Then*

(a) *If* $\{C_\lambda : \lambda \in \Lambda\}$ *is a family of subsets of* A, *then*

$$(1)\quad f\left(\bigcup_{\lambda\in\Lambda} C_\lambda\right) = \bigcup_{\lambda\in\Lambda} f(C_\lambda) \text{ and}$$

$$(2)\quad f\left(\bigcap_{\lambda\in\Lambda} C_\lambda\right) \subseteq \bigcap_{\lambda\in\Lambda} f(C_\lambda).$$

(b) *If* $\{D_\lambda : \lambda \in \Lambda\}$ *is a family of subsets of* B, *then*

$$(1)\quad f^{-1}\left(\bigcup_{\lambda\in\Lambda} D_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{-1}(D_\lambda) \text{ and}$$

$$(2)\quad f^{-1}\left(\bigcap_{\lambda\in\Lambda} D_\lambda\right) = \bigcap_{\lambda\in\Lambda} f^{-1}(D_\lambda).$$

GRAPHS OF FUNCTIONS $f : \mathbb{R} \to \mathbb{R}$

B.2.14 Definition. *The* **graph** *of a function $f : A \to B$, where $A, B \subseteq \mathbb{R}$, is the set of all points (x, y) in the Cartesian (rectangular) coordinate system for which $y = f(x)$. That is,*

$$\text{graph}(f) = \{(x, f(x) : x \in \mathcal{D}(f)\}.$$

Thus, a function $f : A \to B$, where $A, B \subseteq \mathbb{R}$,

- must pass the **vertical line test:**

  (a) no vertical line may intersect its graph in more than one point;

  (b) every vertical line which intersects the set A on the x-axis also intersects its graph.

- is **1-1** iff it passes the **horizontal line test**: no horizontal line may intersect its graph at more than one point.
- is **onto** B iff every horizontal line which intersects B on the y-axis also intersects its graph.

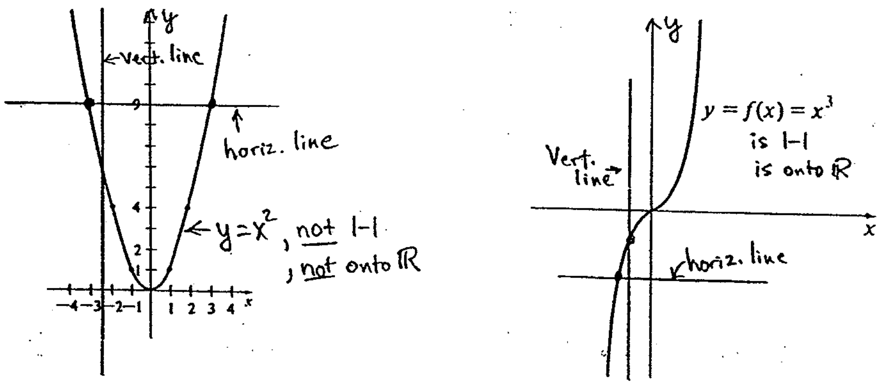

Figure 6

EXERCISE SET B.2

1. For each of the following functions f, find (the largest possible subsets of $\mathbb{R}$ that could be) $\mathcal{D}(f)$ and $\mathcal{R}(f)$, and tell whether or not f is 1-1 and/or onto $\mathbb{R}$.

   (a) $f(x) = 2x - 3$ (b) $f(x) = |x| - 2$

   (c) $f(x) = \sqrt{3x - 4}$ (d) $f(x) = x^2 + 2x + 4$

   (e) $f(x) = \dfrac{|x|}{x}$ (f) $f(x) = \dfrac{1}{x^2 + 1}$

2. Let $f(x) = 4 - x^2$. Find
   - (a) $\mathcal{D}(f)$ (b) $\mathcal{R}(f)$
   - (c) $f[0,1]$ (d) $f^{-1}[0,1]$
   - (e) $f(0,2)$ (f) $f^{-1}(0,4)$
   - (g) $f^{-1}[2,4]$ (h) $f^{-1}[-4,0]$
   - (i) $f^{-1}(0,\infty)$ (j) $f^{-1}(-\infty,2]$
   - (k) $f^{-1}(\{0\})$ (l) $f^{-1}(\{-1\})$

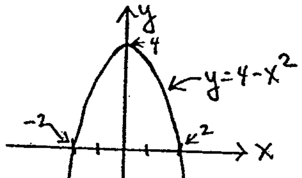

Figure 7

3. Let $f(x) = 2^x$. Find
   - (a) $\mathcal{D}(f)$ (b) $\mathcal{R}(f)$
   - (c) $f[0,1]$ (d) $f(-\infty,2)$
   - (e) $f(0,\infty)$ (f) $f[-1,\frac{1}{2}]$
   - (g) $f^{-1}[1,2]$ (h) $f^{-1}(2,8)$
   - (i) $f^{-1}(0,1)$ (j) $f^{-1}(-\infty,0)$

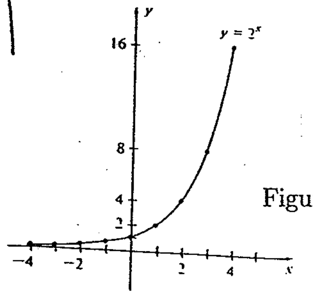

Figure 8

4. Redo Example B.2.12 using the function $f(x) = 4 - x^2$ instead of the function given there.

5. Redo Example B.2.12 using the function $f(x) = 2^x$ instead of the function given there.

6. Prove Theorem B.2.11 (b).

7. Prove Theorem B.2.11 (c).

8. Prove Theorem B.2.11 (d).

9. Prove Theorem B.2.11 (f).

10. Prove Theorem B.2.13 (a) (1).

11. Prove Theorem B.2.13 (a) (2).

12. Prove Theorem B.2.13 (b) (1).

13. Prove Theorem B.2.13 (b) (2).

B.3 ALGEBRA OF REAL-VALUED FUNCTIONS

B.3.1 Definition. *Let $\mathcal{S}$ denote an arbitrary set. Any function $f : \mathcal{S} \to \mathbb{R}$ is called a* ***real-valued function*** *on $\mathcal{S}$. We shall consider the set of all such functions,*

$$\mathcal{F}(\mathcal{S}, \mathbb{R}) = \{\text{all functions } f : \mathcal{S} \to \mathbb{R}\}.$$

On this set $\mathcal{F}(\mathcal{S}, \mathbb{R})$ we can define algebraic operations. In particular, we define

(a) **Addition:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $f+g$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{(f+g)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{f(x)+g(x)}_{\text{in } \mathbb{R}}.$$

(b) **Multiplication by "scalars"** $r \in \mathbb{R}$: $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, and $\forall r \in \mathbb{R}$, we define the function rf by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{(rf)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{r \cdot f(x)}_{\text{in } \mathbb{R}}.$$

(c) **Multiplication** : $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function fg by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{(fg)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{f(x) \cdot g(x)}_{\text{in } \mathbb{R}}.$$

(d) **Division** : $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $\frac{f}{g}$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{\left(\frac{f}{g}\right)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{\left(\frac{f(x)}{g(x)}\right)}_{\text{in } \mathbb{R}}$$

Notice that $\frac{f}{g}$ is not necessarily in $\mathcal{F}(\mathcal{S}, \mathbb{R})$, since we do not know if the denominator is ever 0 without knowing the specific function $g(x)$. The domain of $\frac{f}{g}$ may be different from $\mathcal{S}$.

(e) **Absolute value:** $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $|f|$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{|f|}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{|f(x)|}_{\text{in } \mathbb{R}}.$$

(f) **Maximum:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $\max\{f, g\}$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{\max\{f,g\}}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{\max\{f(x), g(x)\}}_{\text{in } \mathbb{R}}.$$

(g) **Minimum:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $\min\{f, g\}$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{\min\{f,g\}}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{\min\{f(x), g(x)\}}_{\text{in } \mathbb{R}}.$$

B.3.2 Example. *Consider the functions* $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *given by* $f(x) = 3x + 2$ *and* $g(x) = \dfrac{1}{x-1}$. *Then* $\forall x \in \mathbb{R}$,

(a) $(f+g)(x) = f(x) + g(x) = 3x + 2 + \dfrac{1}{x-2}$.

(b) $2f(x) = 2(3x+2) = 6x + 4$.

(c) $(fg)(x) = f(x)g(x) = (3x+2)\left(\dfrac{1}{x-2}\right) = \dfrac{3x+2}{x-2}$.

(d) $|f|(x) = |f(x)| = |3x+2|$.

(e) $\max\{f, g\}(x) = \max\left\{3x+2, \dfrac{1}{x-2}\right\}$; *for example,* $\max\{f, g\}(0) = 2$ *and* $\max\{f, g\}(-1) = -\dfrac{1}{3}$.

(f) $\min\{f, g\}(x) = \min\left\{3x+2, \dfrac{1}{x-2}\right\}$; *for example,* $\min\{f, g\}(0) = -\dfrac{1}{2}$ *and* $\min\{f, g\}(-1) = -1$.

B.3.3 Theorem. ***(Algebra of functions)*** *Let* $\mathcal{S}$ *denote an arbitrary nonempty set. Then* $\mathcal{F}(\mathcal{S}, \mathbb{R})$*, together with the operations (a) - (c) specified in Definition B.3.1 above, satisfies the following properties:*

(a) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f + g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;
(b) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f + (g+h) = (f+g) + h$;
(c) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f + g = g + f$;
(d) $\exists \Theta \in \mathcal{F}(\mathcal{S}, \mathbb{R}) \ni \forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f + \Theta = \Theta + f = f$;
(e) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $\exists -f \in \mathcal{F}(\mathcal{S}, \mathbb{R}) \ni f + (-f) = \Theta$;
(f) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $r \in \mathbb{R}$, $rf \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;
(g) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $\forall\, r \in \mathbb{R}$, $r(f+g) = rf + rg$;
(h) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $\forall\, r, s \in \mathbb{R}$, $(r+s)(f) = rf + sf$;
(i) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $\forall\, r, s \in \mathbb{R}$, $r(sf) = (rs)f = s(rf)$;
(j) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $1f = f$;
(k) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $fg \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;
(l) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f(gh) = (fg)h$;
(m) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $fg = gf$;
(n) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f(g+h) = fg + fh$;

Proof. (a) Let $f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. By Definition B.3.1 (a), $f + g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$.

(b) Let $f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}
[f + (g+h)](x) &= f(x) + (g+h)(x) && \text{by def'n of } f + (g+h) \\
&= f(x) + [g(x) + h(x)] && \text{by def'n of } g + h \\
&= [f(x) + g(x)] + h(x)] && \text{by axiom (A2) of } \mathbb{R} \\
&= (f+g)(x) + h(x) && \text{by def'n of } f + g \\
&= [(f+g) + h](x) && \text{by def'n of } (f+g) + h
\end{aligned}$$

Thus, by Definition B.2.2, $f + (g+h) = (f+g) + h$.
(c) Exercise 5.
(d) Define the function $\Theta \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ by the rule

$$\forall x \in \mathcal{S},\ \Theta(x) = 0 \in \mathbb{R}.$$

Let $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}(\Theta + f)(x) &= \Theta(x) + f(x) && \text{by def'n of } \Theta + f\\ &= 0 + f(x) && \text{by def'n of the } \Theta \text{ function}\\ &= f(x) && \text{by axiom (A3) of } \mathbb{R}\end{aligned}$$

Thus, by Definition B.2.2, $\Theta + f = f$. Also, $f + \Theta = f$, by part (c) above.

(e) Exercise 6.

(f) Exercise 7.

(g) Exercise 8.

(h) Let $f, \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ and $r, s \in \mathbb{R}$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}[(r+s)f](x) &= (r+s) \cdot f(x) && \text{by def'n of } (r+s)f\\ &= r \cdot f(x) + s \cdot f(x) && \text{by axiom (D) of } \mathbb{R}\\ &= (rf)(x) + (sf)(x) && \text{by def'n of } rf \text{ and } sf\\ &= (rf + sf)(x) && \text{by def'n of } rf + sf\end{aligned}$$

Thus, by Definition A.6.2, $(r + s)f = fr + sf$.

(i) Exercise 9.

(j) Exercise 10.

(k) Exercise 11.

(l) Let $f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}[f(gh)](x) &= f(x) \cdot (gh)(x) && \text{by def'n of } f(gh)\\ &= f(x) \cdot [g(x) \cdot h(x)] && \text{by def'n of } gh\\ &= [f(x) \cdot g(x)] \cdot h(x) && \text{by axiom (M2) of } \mathbb{R}\\ &= (fg)(x) \cdot h(x) && \text{by def'n of } fg\\ &= [(fg)h](x) && \text{by def'n of } (fg)h\end{aligned}$$

Thus, by Definition B.2.2, $f(gh) = (fg)h$.

(m) Exercise 12.

(n) Exercise 13. ■

Students who have had a course in linear algebra will observe that Properties (a) - (j) say that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition and multiplication by "scalars" in Definition B.3.1, is a **vector space**. Students who have also had a course in abstract algebra will observe that properties (a) - (e) and (k) - (n) say that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition and multiplication in Definition B.3.1, is a **commutative ring**. They may be interested in proving that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition and multiplication in Definition B.3.1, is not an integral domain.

Properties (a) - (n) taken together, say that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition, multiplication by "scalars", and multiplication in Definition B.3.1, is a **commutative algebra**. This may perhaps be a new algebraic term for you.

COMPOSITE FUNCTIONS AND INVERSES

B.3.4 Definition. *If* $f : A \to B$ *and* $g : B \to C$, *then the* **composite function** $g \circ f$ *is defined by the rule*

$$(g \circ f)(x) = g(f(x)).$$

The following schematic diagram may be helpful in giving an intuitive understanding of $g \circ f$:

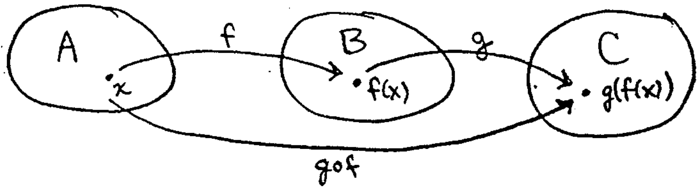

Figure 9

BEWARE! Notice the reversal of orientation: in the schematic drawing, the function f is drawn to the left of g, but in the composite function notation $g \circ f$, g is written to the left of f. When the composite function $g \circ f$ operates on the element x, f operates first and then g operates on the result, despite the fact that when we write "$g \circ f$" we write g first. Care must be exercised to avoid confusion.

ALSO, BEWARE: In general,

$$f \circ g \neq g \circ f,$$

although sometimes they are equal.

B.3.5 Example. *For the functions* $f(x) = 3x + 2$ *and* $g(x) = \dfrac{1}{x-2}$,

$$(g \circ f)(x) = g(3x - 2) = \frac{1}{(3x+2)-2} = \frac{1}{3x}, \text{ whereas}$$
$$(f \circ g)(x) = f\left(\frac{1}{x-2}\right) = 3\left(\frac{1}{x-2}\right) + 2 = \frac{2x-1}{x-2}.$$

In this example, $f \circ g \neq g \circ f$.

B.3.6 Theorem. *Composite functions obey the associative law. That is, if* $f : A \to B$, $g : B \to C$, *and* $h : C \to D$, *then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof. Suppose $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then $\forall x \in A$,

$$\begin{aligned}
[h\circ(f\circ g)](x) &= h[(g\circ f)(x)] && \text{by def'n of } h\circ(g\circ f)\\
&= h[g(f(x))] && \text{by def'n of } g\circ f\\
&= (h\circ g)[f(x)] && \text{by def'n of } h\circ g\\
&= [(h\circ g)\circ f](x) && \text{by def'n of } (h\circ g)\circ f
\end{aligned}$$

Thus, by Definition B.2.2, $h\circ(g\circ f) = (h\circ g)\circ f$. ■

B.3.7 Theorem. *Suppose $f : A \to B$ and $g : B \to C$.*
(a) If f and g are both 1-1, then so is $g\circ f$.
(b) If f and g are both onto, then so is $g\circ f$.
(c) If f and g are both 1-1 correspondences, then so is $g\circ f$.

Proof. Exercise 14. ■

B.3.8 Definition. *Let A denote an arbitrary set. The* **identity function on** *A is the function $i_A : A \to A$ defined by the rule*

$$\forall x \in A,\ i_A(x) = x.$$

Note: The identity function is a 1-1 correspondence.

B.3.9 Theorem. *Let A and B denote an arbitrary sets. The identity function on A, $i_A : A \to A$ and $i_B : B \to B$ satisfy the following rules:*
(a) $\forall f : A \to B,\ f\circ i_A = f$;
(b) $\forall g : B \to A,\ i_B\circ g = g$.

Proof. Exercise 15. ■

B.3.10 Theorem. *Suppose $f : A \to B$.*
(a) If $\exists g : B \to C \ni g\circ f$ is 1-1, then f is 1-1.
(b) If $\exists h : D \to A \ni f\circ h$ is onto B, then f is onto B.

Proof. (a) Suppose $\exists g : B \to C \ni g\circ f$ is 1-1. Let $a \neq a'$ in A. Since $g\circ f$ is 1-1,

$$\begin{aligned}
(g\circ f)(a) &\neq (g\circ f)(a'); \quad \text{which means that}\\
g(f(a)) &\neq g(f(a')).
\end{aligned}$$

This cannot happen unless $f(a) \neq f(a')$. Therefore, f is 1-1.

(b) Suppose $\exists h : D \to A \ni f\circ h$ is onto B. Let $b \in B$. Since $f\circ h$ is onto B, $\exists d \in D \ni (f\circ h)(d) = b$. Let $a = h(d)$. Then $a \in A$ and $f(a) = f(h(d)) = b$. That is, $\forall\, b \in B$, $b \in \mathcal{R}(f)$. Therefore, f is onto B. ■

B.3.11 Definition. *Suppose $f : A \to B$. If $\exists$ function $g : B \to A \ni g\circ f = i_A$ and $f\circ g = i_B$, then we say that f is* **invertible**, *and we say that the function g is the* **inverse function** *of f. In symbols,*

$$g = f^{-1}.$$

B.3.12 Theorem. *A function $f : A \to B$ is* **invertible** *iff f is 1-1 and onto B (that is, f is a 1-1 correspondence). Moreover, if $g = f^{-1}$, then $f = g^{-1}$.*

Proof. First, the $\Rightarrow$ direction. Suppose $f : A \to B$ is invertible. Then $\exists$ function $g : B \to A \ni g \circ f = i_A$ and $f \circ g = i_B$. Since i_A is 1-1, Theorem B.3.10 (a) says that f is 1-1. Since i_B is onto B, Theorem B.3.10 (b) says that f is onto B.

Next, the $\Leftarrow$ direction. Suppose $f : A \to B$ is 1-1 and onto B. Let $b \in B$. Since f is onto B, $\exists a \in A \ni f(a) = b$. Moreover, since f is 1-1, there is no more than one such a. Define

$$g(b) = a.$$

Then $\forall a \in A$, $(g \circ f)(a) = g(f(a)) = g(b) = a = i_A(a)$. Also, $\forall b \in B$, $(f \circ g)(b) = f(g(b)) = f(a) = b = i_B(b)$. That is, $g \circ f = i_A$ and $f \circ g = i_B$. Therefore, by Definition B.3.11, f is invertible.

Finally, the proof of $g = f^{-1} \Rightarrow f = g^{-1}$ is Exercise 16. ■

B.3.13 Corollary. *If $f : A \to B$ is a 1-1 correspondence, so is $f^{-1} : B \to A$*

Proof. Apply Theorem B.3.10. ■

EXERCISE SET B.3

1. Let $f(x) = 2x+1$ and $g(x) = x^2-1$. Find $(f+g)(x)$, $(f-g)(x)$, $f(x+2)$, $f(x)+2$, $g(x+2)$, $g(x)+2$, $3f(x)$, $f(3x)$, $3g(x)$, $g(3x)$, $(fg)(x)$, $\left(\dfrac{f}{g}\right)(x)$, $|f|(x)$, $\max\{f,g\}(x)$, $\min\{f,g\}(x)$, $(f \circ g)(x)$, and $(g \circ f)(x)$.

2. Repeat Exercise 1 with $f(x) = \dfrac{x}{x-2}$ and $g(x) = \dfrac{3}{x}$.

3. For each of the following functions, f, find $\mathcal{D}(f)$ and $\mathcal{R}(f)$, and tell whether f is 1-1:

   (a) $f(x) = 7x+8$
   (b) $f(x) = \sqrt{x+1}$
   (c) $f(x) = \sqrt{x^2-1}$
   (d) $f(x) = \ln x$
   (e) $f(x) = e^x$
   (f) $f(x) = \dfrac{x}{x+1}$
   (g) $f(x) = \sin x$
   (h) $f(x) = x^3+2$

4. Which of the functions given in Exercise 3, viewed as $f : \mathcal{D}(f) \to \mathcal{R}(f)$, are invertible. Find f^{-1} where possible.

5. Prove Theorem B.3.3 (c).

6. Prove Theorem B.3.3 (e).

7. Prove Theorem B.3.3 (f).

8. Prove Theorem B.3.3 (g).
9. Prove Theorem B.3.3 (i).
10. Prove Theorem B.3.3 (j).
11. Prove Theorem B.3.3 (k).
12. Prove Theorem B.3.3 (m).
13. Prove Theorem B.3.3 (n).
14. Prove Theorem B.3.7.
15. Prove Theorem B.3.9.
16. Finish proving Theorem B.3.12, by proving that $g = f^{-1} \Rightarrow f = g^{-1}$.