Group Presentations and Cayley Graphs

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Abstract

It was conjectured about 40 years ago that every connected Cayley graph has a hamiltonian cycle. This is easy to prove for Cayley graphs on abelian groups, but we are nowhere near a proof of the general case. The talk will discuss some of the progress that has been made, and some of the many open problems.

Groups

Definition

Let $*$ be a binary operation defined on G. G is called a group if

1. $*$ is associative.
2. G has an element e such that $x * e = e * x = x$ for any $x \in G$. Such an element e is called the identity of G.
3. Each element $a \in G$ has an inverse. That is, there is an element $b \in G$ such that $a * b = b * a = e$. b is called the inverse of a, and denoted by a^{-1}.

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Example:

- Let $n \geq 1$ be an integer. $\mathbb{Z}_n$ is an abelian group.
- $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$ is an abelian group.

Cyclic subgroups and generating sets

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Let G be a group and $a \in G$. Then $\langle a \rangle = \{a^i | i \in \mathbb{Z}\}$ is called a cyclic subgroup with the generator a. If $G = \langle a \rangle$, then G is called a cyclic group with a generator a.

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- A generating set of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $X = \{(1, 0), (0, 1)\}$.
- For the group $\mathbb{Z}_n$, a nonempty set of integers in $\mathbb{Z}_n$ is a generating set if and only if its gcd is 1. For instance, the set $X = \{4, 7\}$ generates $\mathbb{Z}_{24}$, but $\{6, 9\}$ does not generate $\mathbb{Z}_{24}$.

Permutation groups

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Remark: Let $\alpha = (1, 2, 3)$ and $\beta = (1, 3)$ be permutations in S_3. Then $\alpha\beta = (2, 3)$ and $\beta\alpha = (1, 2)$. So S_n is not abelian.

Dihedral group D_n

Definition

Let $a = (1, 2, \cdots, n)$ and

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & i & \cdots & n-1 & n \\ 1 & n & n-1 & n-2 & n-3 & \cdots & n+2-i & \cdots & 3 & 2 \end{pmatrix}$$

be permutations in S_n. Then the subgroup generated by a and b is called the dihedral group of degree n and is denoted by D_n.

Action of a group

Definition

An action of a group G on a set S is a function $G \times S \mapsto S$ (usually denoted by $(g, x) \longmapsto gx$) such that for all $x \in S$ and $g_1, g_2 \in G$,

$$ex = x, \qquad \text{and} \qquad (g_1 g_2)x = g_1(g_2)x$$

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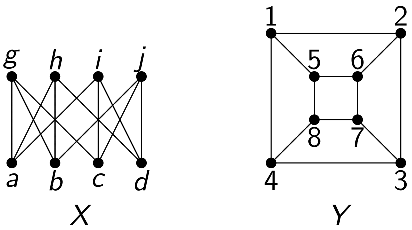
Definition

Let G be a group acting on S containing at least two elements. If for any given $x, y \in S$, there exists $g \in G$ such that $gx = y$, then G is transitive.

Isomorphisms of graphs

Definition

Let X and Y be graphs. Two graphs X and Y are isomorphic if there is a bijection $\phi : V(X) \to V(Y)$ such that $xy \in E(X)$ if and only if $\phi(x)\phi(y) \in E(Y)$. The function ϕ is called an isomorphism from X to Y.



$\phi(a) = 1$
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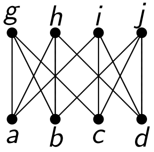
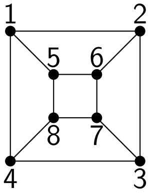


X



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Remark: If $X = Y$, ϕ is also called an automorphism of X. Thus, an automorphism is a permutation of the vertices of X that maps edges to edges and nonedges to nonedges.

Automorphism group

Remark: Consider the set of all automorphisms of a graph X. Then

1. The identity permutation is an automorphism. We denote it by e.
2. If g is an automorphism of X, then so its inverse g^{-1}.
3. If f and g are automorphisms of X, then gh is an automorphism.

So the set of all automorphisms of X forms a group. This group is called the automorphism group of X, and is denote by $Aut(X)$.

Vertex-transitive graphs

Definition

A graph X is called vertex transitive if its automorphism group $Aut(X)$ acts transitively on $V(X)$. Thus for any two distinct vertices of X, there is an automorphism mapping one to the other. Informally speaking, a graph is vertex-transitive if every vertex has the same local environment, so that no vertex can be distinguished from any other based on the vertices and edges surrounding it.

Vertex-transitive graphs

Example: Let X be a transitive graph.

- If $|V(X)| = 1$, then X is a single-vertex graph.

- If $|V(X)| = 2$, then X is one of the following.

- If $|V(X)| = 3$, then X is one of the following.

Vertex-transitive graphs

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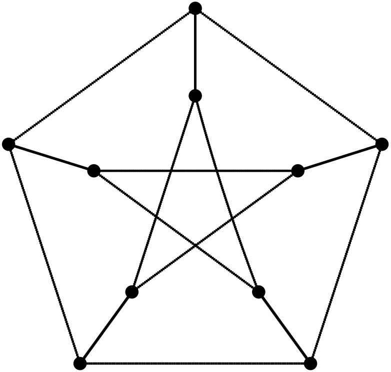
- The Petersen graph is vertex transitive.



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Cayley subset

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Let G be a group with the identity 1. If S is a subset of G satisfying

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- If $G = A_4$, then $S = \{(1, 2, 3), (1, 3, 2), (1, 2)(3, 4)\}$ is a Cayley subset of G.

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- If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $S = \{(0,1), (1,0)\}$ is a Cayley subset of G.

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- The Cayley graph $Cay(G, S)$ is the undirected graph with vertex set G and edge set containing an edge from g to gs and from g to gs^{-1} whenever $g \in G$ and $s \in S$.

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- The subset S contains the identity of G if and only if the Cayley graph $Cay(G, S)$ contains a self-loop at every vertex.

Cayley digraph

Definition

The Cayley digraph $\overrightarrow{Cay}(G,S)$ on a group G with a subset S is the digraph with vertex set G and arc set containing an arc (g,s) from g to gs whenever $g \in G$ and $s \in S$.

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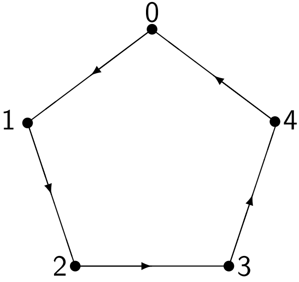

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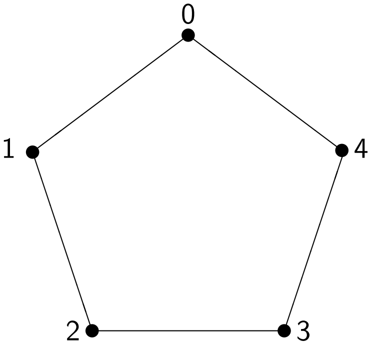

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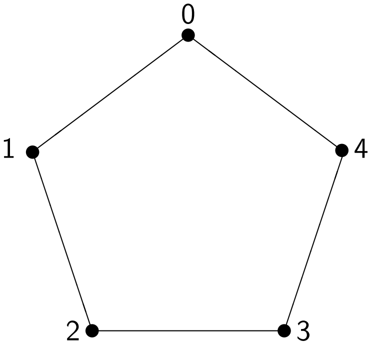

Remark: Generally, $Cay(\mathbb{Z}_n, \{1, -1\})$ is the cycle C_n.

$Cay(S_3, \{(1,2,3),(1,3,2)\})$

Let $G = S_3 = \{(1),(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}$ and $S = \{(1,2,3),(1,2,3)^{-1}\} = \{(1,2,3),(1,3,2)\}$.

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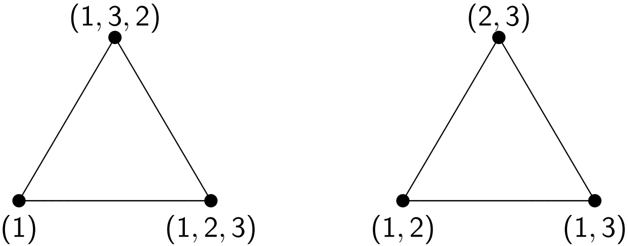

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- $Cay(G, S)$ is connected if and only if S generates G.
- $Cay(G, S)$ is vertex transitive.

  Proof. For each $g \in G$ the mapping

  $$\rho_g : x \to xg$$

  is a permutation of the elements of G. This is an automorphism of $Cay(G, S)$ because

  $$(yg)(xg)^{-1} = ygg^{-1}x^{-1} = yx^{-1}$$

  and so $xg \sim yg$ if and only if $x \sim y$. The permutation ρ_g form a subgroup of the automorphism group of $Cay(G, S)$ isomorphic to G. This subgroup acts transitively on the vertices of $Cay(G, S)$ because for any two vertices g and h, the automorphism $\rho_{g^{-1}h}$ maps g to h.

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  Proof. There are only two groups of order 10, the cyclic group $\mathbb{Z}_{10}$ and the dihedral group D_{10}. None of the cubic Cayley graphs on these groups are isomorphic to the Peterson graph.

Motivation

Today studying the Hamiltonian property of graphs is a favorite problem for graph and group theorists. Hamiltonian paths and cycles play an important role in computer science and in combinatorial designs. For example, it is well-known fact that the Hamiltonian property of the hypercube H_n is demonstrated by Gray codes. Testing whether a graph is Hamiltonian is one of the classical NP-complete problems.

Cayley graphs with involutions

Theorem (Rapaport-Strasser)

Let G be a finite group, generated by three involutions α, β, γ *such that* $\alpha\beta = \beta\alpha$*. Then the Cayley graph* $Cay(G, \{\alpha, \beta, \gamma\})$ *has a hamiltonian cycle.*

An element $\alpha \in G$ is called an involution if $\alpha^2 = 1$.

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Theorem (Rankin)

Let G be a finite group, generated by two elements α, β *such that* $(\alpha\beta)^2 = 1$*. Then the Cayley graph* $Cay(G, \{\alpha, \beta\})$ *has a hamiltonian cycle.*

Open problem

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To be more precisely he stated a research problem in 1970 asking how one can construct "a finite connected undirected graph which is symmetric and has no simple path containing all the vertices."

- A graph is symmetric if for any two vertices x and y it has an automorphism mapping x onto y.

Conjecture

There are only four vertex-transitive graphs on more than two vertices which do not have a hamiltonian cycle, and all of these graphs have a hamiltonian path. They are

- the Petersen graph
- the Coxeter graph
- the graphs obtained from each of these two graphs by replacing each vertex with a triangle and joining the vertices in a natural way.

In particular, it is unknown of a vertex-transitive graph without a hamiltonian path. Furthermore, it was noted that all of the above four graphs are not Cayley graphs.

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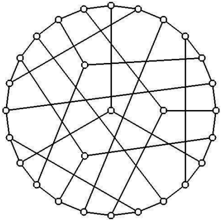
Remark: No much progress.

Coxeter graph



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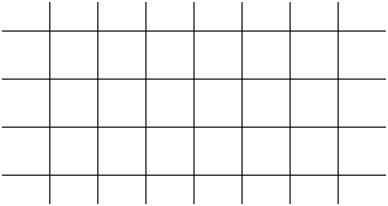

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In 1996, Babai made the following conjecture.

Conjecture (Babai)

For some $\varepsilon > 0$*, there exists infinitely many connected vertex transitive graphs (even Cayley graphs)* X *without cycles of length* $\geq (1 - \varepsilon)|X|$.

Group of special orders

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Theorem (Jungreis et al.)

Let p and q be primes. Then every Cayley graph on a group of order $pq, 4q(q > 3), p^2q(2 < p < q), 2p^2, 2pq, 8p$*, or* $4p^2$ *is hamiltonian.*

Group of special orders

Theorem (Witte et.al)

If $Cay(G, S)$ is a connected Cayley graph with n vertices, and the prime factorization of n is very small, then $Cay(G, S)$ has a hamiltonian cycle. More precisely, if p, q, and r are distinct primes, then n can be of the form kp with $k < 23$ and $k \neq 16$, or of the form kpq with $k < 6$, or of the form pqr, or of the form kp^2 with $k < 5$, or of the form kp^3 with $k < 3$.

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A rare positive result for all finite groups was obtained in 2004 by Pak, Radoičić.

Theorem (Pak, Radoičić)

Every finite group G of size $|G| \geq 3$ has a generating set S of size $|S| \leq \log_2 |G|$ such that the corresponding Cayley graph $Cay(G, S)$ has a hamiltonian cycle.

Abelian groups

Theorem (Marušič)

A Cayley graph $Cay(G, S)$ of an abelian group G with hat least three vertices has a hamiltonian cycle.

Symmetric groups

There are also some results for Cayley graphs on the symmetric group S_n generated by transpositions. These graphs have been proposed as models for the design and analysis of interconnection networks. Moreover, hamiltonian paths in Cayley graphs on S_n provide an algorithm for creating the elements of S_n from a particular generating set.

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Theorem

$Cay(G, S)$ has a hamiltonian cycle if G is dihedral and $4 \mid |G|$.

Open problems

Problem

Let S be a generating set for S_n in which every element has order 2. Is the graph $Cay(S_n, S)$ hamiltonian?

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Problem

Does $Cay(G, S)$ have a hamiltonian cycle if G is dihedral?

Direct products

In their examination of Cayley graphs of groups of low order, Jungreis et al. obtained several results on direct products of special forms.

Theorem (Jungreis et al.)

Every Cayley graph on a group of the form $\mathbb{Z}_p \times A_4$ *(where* p *is prime, and* A_4 *is the alternating group of degree 4) is hamiltonian.*

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Problem

Find hamiltonian cycle if $G = P \times Q$, where $|P|$ and $|Q|$ are prime powers.

Factorization

- A factor of a graph G is a spanning subgraph, i.e., a subgraph that has the same vertex set as G.
- A k-factor of a graph is a spanning k-regular subgraph.
- A k-factorization partitions the edges of the graph into disjoint k-factors.
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- A k-factorization partitions the edges of the graph into disjoint *k*-factors.
- A 1-factorization of a graph is a partition of the edge set into 1-factors.

Theorem

A given connected Cayley graph on the group G has a 1-factorization if one of the following holds:

1. $|G| = 2^k$ *for an integer* k*;*
2. G *is an even odd abelian group; or*
3. G *is dihedral or dicyclic.*

Hamiltonian decomposition

Definition

We say that X is hamiltonian decomposable if

- either X is $2k$-regular and $E(G)$ can be partitioned into k hamiltonian cycles, or
- or X is $2k+1$-regular and $E(G)$ can be partitioned into k hamiltonian cycles and a 1-factor.

Conjecture

A recent survey on hamiltonian decompositions of graphs was given by Alspach et al.. Bermond had given an earlier survey on hamiltonian decompositions of graphs. The following problem was posed by Alspach.

Conjecture (Alspach)

Does every connected Cayley graph on a finite abelian group have a hamiltonian decomposition?

1. The answer is trivially true when the degree of the graph is 2.
2. When the degree of the graph is 3, the answer is yes because every such graph has a hamiltonian cycle.
3. When the degree is 4, Bermond et al. showed that every 4-regular Cayley graph on an abelian group is hamiltonian decomposable.
4. When the degree is 5, Alspach et al. showed that every 5-regular Cayley graph on an abelian group is hamiltonian decomposable.

Hamiltonian decomposition

Definition

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Theorem (Li)

If $Cay(G, S)$ is a connected Cayley graph on an abelian group G and S is a minimal generating Cayley set, then G has a hamiltonian decomposition.

Miscellaneous

A graph G is hamiltonian connected if every two vertices of G are connected by a Hamiltonian path.

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Problem (Chen and Quimpo)

Characterize the hamiltonian connected Cayley graphs of order pq, where p and q are primes.

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Isomorphisms of finite Cayley graphs

Some of the most interesting and deepest work on Cayley graphs has revolved around the question of trying to determine when two Cayley graphs are isomorphic.

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Problem

Given two Cayley graphs $Cay(G, S)$ *and* $Cay(H, T)$*, determine whether or not* $Cay(G, S) \cong Cay(H, T)$.

CI-graph

We consider the isomorphism problem of Cayley graphs over the same group. Let G be a group, and let $\Gamma = Cay(G, S)$ for some subset $S \subseteq G$. Let π be an automorphism of G. Let $T = \pi(S)$. Then it is easily shown that π induces an isomorphism from $Cay(G, S)$ to the Cayley graph $Cay(G, T)$. Such an isomorphism is called a Cayley isomorphism. However, it is of course possible for two Cayley graphs $Cay(G, S)$ and $Cay(G, T)$ to be isomorphic but no Cayley isomorphisms mapping S to T. Here we investigate the conditions under which $Cay(G, S) \cong Cay(G, T)$ if and only if $\pi(S) = T$ for some $\pi \in Aut(G)$.

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Definition

A Cayley graph $Cay(G, S)$ is called a CI-graph if whenever $Cay(G, S) \cong Cay(G, S')$, there exists an automorphism $\pi \in Aut(G)$ such that $S' = \pi(S)$. A group G is called a CI-group if every Cayley graph on G is a CI-graph.

Open problem

One long-standing open question about Cayley graphs is as follows:

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Which Cayley graphs for a group G are CI-graphs?

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Remark: H. Cai, On isomorphisms of finite Cayley graphs – a survey, Discrete Mathematics 256 (2002) 301-334.

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Thank You!