Z_3-connectivity in Abelian Cayley Graphs

Hong-Jian Lai, Mingquan Zhan*, Taoye Zhang, Ju Zhou
Millersville University of Pennsylvania

July 10, 2011

- G:= a graph, with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \cdots, e_m\}$.
- $N_G(v)$:= the neighborhood of v in G.
- $D(G)$:= an orientation of G.
- Let $v \in D(G)$.
  - $E^+(v)$: the set of all edges with tails at v.
  - $E^-(v)$: the set of all edges with heads at v.

Notations

- A:= an Abelian (additive) group with identity 0, and with $|A| \geq 3$, $A^* = A - \{0\}$.

Notations

- A:= an Abelian (additive) group with identity 0, and with $|A| \geq 3$, $A^* = A - \{0\}$.
- $F(G, A) = \{f \mid f : E(G) \mapsto A\}$, $F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}$.

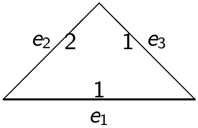
- A:= an Abelian (additive) group with identity 0, and with $|A| \geq 3$, $A^* = A - \{0\}$.
- $F(G, A) = \{f \mid f : E(G) \mapsto A\}$, $F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}$.
- A function $f : E \mapsto A$ can be viewed as an m-dimensional vector

$$f = (f(e_1), f(e_2), \cdots, f(e_m)).$$

- A:= an Abelian (additive) group with identity 0, and with $|A| \geq 3$, $A^* = A - \{0\}$.
- $F(G, A) = \{f \mid f : E(G) \mapsto A\}$, $F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}$.
- A function $f : E \mapsto A$ can be viewed as an m-dimensional vector

$$f = (f(e_1), f(e_2), \cdots, f(e_m)).$$

Example: For $A = Z_3 = \{0, 1, 2\}$, $f = (f(e_1), f(e_2), f(e_3)) = (1, 2, 1)$.



- Assumption: For any graph G, we assume that a fixed orientation $D(G)$ of G is given.

- Assumption: For any graph G, we assume that a fixed orientation $D(G)$ of G is given.

- For a function $f \in F^*(G,A)$, the boundary of f is a function $\partial f : V(G) \mapsto A$ such that for each vertex $v \in V(G)$, $\partial f(v)$ equals to the total amount of f-values on the edges directed out from v minus the total amount of f-values on the edges directed into v:

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

- Assumption: For any graph G, we assume that a fixed orientation $D(G)$ of G is given.

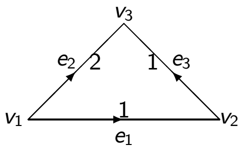
- For a function $f \in F^*(G, A)$, the boundary of f is a function $\partial f : V(G) \mapsto A$ such that for each vertex $v \in V(G)$, $\partial f(v)$ equals to the total amount of f-values on the edges directed out from v minus the total amount of f-values on the edges directed into v:

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

Example: For $A = Z_3 = \{0, 1, 2\}$,

$$\partial f = (\partial f(v_1), \partial f(v_2), \partial f(v_3)) = (3, 0, 0) = (0, 0, 0).$$



- An A-nowhere-zero flow (abbreviated as A-NZF) in G is an $f \in F^*(G, A)$ such that $\partial f = \overrightarrow{0}$.

- An A-nowhere-zero flow (abbreviated as A-NZF) in G is an $f \in F^*(G, A)$ such that $\partial f = \overrightarrow{0}$.
- A $\mathbb{Z}$-nowhere-zero flow f is a nowhere-zero k-flow if for any $e \in E(G)$, $0 < |f(e)| < k$.

- An A-nowhere-zero flow (abbreviated as A-NZF) in G is an $f \in F^*(G, A)$ such that $\partial f = \overrightarrow{0}$.
- A $\mathbb{Z}$-nowhere-zero flow f is a nowhere-zero k-flow if for any $e \in E(G)$, $0 < |f(e)| < k$.
- If some orientation $D(G)$ has an A-NZF or a k-NZF, then for any orientation of G also has the same property, and so having an A-NZF or a k-NZF is independent of the choice of the orientation.

- (Tutte's Conjecture) Every 4-edge-connected graph has a nowhere-zero 3-flow.

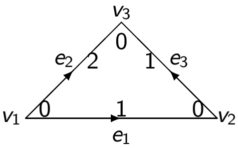
3-Flow Conjecture

- (Tutte's Conjecture) Every 4-edge-connected graph has a nowhere-zero 3-flow.
- The 3-flow conjecture has been verified for a number of infinite classes of graphs, including
  - the projective-planar graphs,
  - Cartesian product graphs,
  - locally connected graphs,
  - the squares of graphs.

  Furthermore, the 3-flow conjecture holds for random graphs, and can be reduced to 5-edge-connected 5-regular graphs.

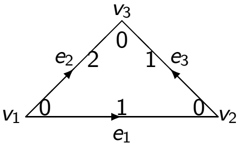
- Define a function $b : V \mapsto A$. So, b can be viewed as an n-dimensional vector $b = (b(v_1), b(v_2), \cdots, b(v_n))$.

Example: $b = (b(v_1), b(v_2), b(v_3)) = (0, 0, 0)$.



- Define a function $b : V \mapsto A$. So, b can be viewed as an n-dimensional vector $b = (b(v_1), b(v_2), \cdots, b(v_n))$.

  Example: $b = (b(v_1), b(v_2), b(v_3)) = (0, 0, 0)$.



- For $f \in F(G, A)$ and $b = \partial f$, we have

$$\sum_{v \in V(G)} b(v) = \sum_{v \in V(G)} \partial f(v) = 0$$

- Any $b : V \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$ is an A-zero-sum function. The set of all A-zero-sum functions is $Z(G, A)$. Obviously, if $b = \overrightarrow{0}$, $b \in Z(G, A)$.

- Any $b : V \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$ is an A-zero-sum function. The set of all A-zero-sum functions is $Z(G, A)$. Obviously, if $b = \overrightarrow{0}$, $b \in Z(G, A)$.
- For a given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ with $\partial f = b$ is called an (A, b)-NZF of G. So, if $b = \overrightarrow{0}$, then an (A, b)-NZF of G is same as an A-NZF of G.

- Any $b : V \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$ is an A-zero-sum function. The set of all A-zero-sum functions is $Z(G, A)$. Obviously, if $b = \overrightarrow{0}$, $b \in Z(G, A)$.
- For a given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ with $\partial f = b$ is called an (A, b)-NZF of G. So, if $b = \overrightarrow{0}$, then an (A, b)-NZF of G is same as an A-NZF of G.
- An undirected graph G is A-connected if G has an orientation $D(G)$ such that for every function $b \in Z(G, A)$, G has an (A, b)-NZF.

- Any $b : V \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$ is an A-zero-sum function. The set of all A-zero-sum functions is $Z(G, A)$. Obviously, if $b = \overrightarrow{0}$, $b \in Z(G, A)$.
- For a given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ with $\partial f = b$ is called an (A, b)-NZF of G. So, if $b = \overrightarrow{0}$, then an (A, b)-NZF of G is same as an A-NZF of G.
- An undirected graph G is A-connected if G has an orientation $D(G)$ such that for every function $b \in Z(G, A)$, G has an (A, b)-NZF.
- Define

  $\Lambda_g(G) = \min\{k : G \text{ is } A\text{-connected for every abelian group A with} |A| \geq k\}$.

- Any $b : V \mapsto A$ with $\sum\limits_{v \in V(G)} b(v) = 0$ is an A-zero-sum function. The set of all A-zero-sum functions is $Z(G, A)$. Obviously, if $b = \overrightarrow{0}$, $b \in Z(G, A)$.
- For a given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ with $\partial f = b$ is called an (A, b)-NZF of G. So, if $b = \overrightarrow{0}$, then an (A, b)-NZF of G is same as an A-NZF of G.
- An undirected graph G is A-connected if G has an orientation $D(G)$ such that for every function $b \in Z(G, A)$, G has an (A, b)-NZF.
- Define

  $\Lambda_g(G) = \min\{k : G \text{ is } A\text{-connected for every abelian group A with} |A| \geq k\}.$

- Define $\langle A \rangle = \{G : G \text{ is A-connected}\}$.

- (Jeager et al., 1992) If G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$.

Some Results

- (Jeager et al., 1992) If G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$.
- (Jeager et al., 1992) If G is a 4-edge-connected graph, then $\Lambda_g(G) \leq 4$.

- (Jeager et al., 1992) If G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$.
- (Jeager et al., 1992) If G is a 4-edge-connected graph, then $\Lambda_g(G) \leq 4$.
- (H.-J. Lai, 1998) For the n-cycle C_n, $\Lambda_g(C_n) = n + 1$.

- (Jeager et al., 1992) If G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$.
- (Jeager et al., 1992) If G is a 4-edge-connected graph, then $\Lambda_g(G) \leq 4$.
- (H.-J. Lai, 1998) For the n-cycle C_n, $\Lambda_g(C_n) = n + 1$.
- (Devos et al., 2006) The n-wheel W_n is the graph obtained from a n-cycle by adding a new vertex, called the center of the wheel, and connecting it to every vertex of the n-cycle. Then the wheel $W_{2n}(n \geq 1)$ is Z_3-connected.

- (Jeager et al., 1992) If G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$.
- (Jeager et al., 1992) If G is a 4-edge-connected graph, then $\Lambda_g(G) \leq 4$.
- (H.-J. Lai, 1998) For the n-cycle C_n, $\Lambda_g(C_n) = n + 1$.
- (Devos et al., 2006) The n-wheel W_n is the graph obtained from a n-cycle by adding a new vertex, called the center of the wheel, and connecting it to every vertex of the n-cycle. Then the wheel $W_{2n}(n \geq 1)$ is Z_3-connected.
- (Lai et al., 2011) Let G be a graph and let $v \in V(G)$ be a vertex of degree two. Then G is Z_3-connected if and only if $G - v$ is Z_3-connected.

- (Jeager et al., 1992) If G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$.
- (Jeager et al., 1992) If G is a 4-edge-connected graph, then $\Lambda_g(G) \leq 4$.
- (H.-J. Lai, 1998) For the n-cycle C_n, $\Lambda_g(C_n) = n + 1$.
- (Devos et al., 2006) The n-wheel W_n is the graph obtained from a n-cycle by adding a new vertex, called the center of the wheel, and connecting it to every vertex of the n-cycle. Then the wheel $W_{2n}(n \geq 1)$ is Z_3-connected.
- (Lai et al., 2011) Let G be a graph and let $v \in V(G)$ be a vertex of degree two. Then G is Z_3-connected if and only if $G - v$ is Z_3-connected.
- (Lai, 2000) Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N_G(v) = \{v_1, v_2, \cdots, v_m\}$ and let $W = \{vv_1, vv_2\}$. The graph $G_{[v,W]}$ is obtained from $G - W$ by adding a new edge that joins v_1 and v_2. If $G_{[v,W]} \in \langle A \rangle$, then $G \in \langle A \rangle$.

Some Results

- $K_1 \in \langle A \rangle$.
- (Lai, 2000) For a subset $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the edges in X. If $H \subseteq G$ and $H, G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.

Conjecture (Jeager et al., 1992) If G is a 5-edge-connected graph, then $\Lambda_g(G) = 3$.

Conjecture (Jeager et al., 1992) If G is a 5-edge-connected graph, then $\Lambda_g(G) = 3$.

By Jeager's Conjecture, every connected vertex-transitive graph of degree at least 5 should be Z_3-connected (therefore, should have a nowhere-zero 3-flow). Among vertex-transitive graphs, Cayley graphs are best understood, and therefore are usually the first to be investigated.

Definition

Let Γ be a group with the identity 1. If S is a subset of Γ satisfying

1. $1 \notin S$
2. if $s \in S$, then $s^{-1} \in S$.

then S is called a Cayley subset of Γ.

Definition

Let Γ be a group with the identity 1. If S is a subset of Γ satisfying

1. $1 \notin S$
2. if $s \in S$, then $s^{-1} \in S$.

then S is called a Cayley subset of Γ.

Example:

- Let $\Gamma = \mathbb{Z}_9$. Then $S = \{1, 3, 6, 8\}$ is a Cayley subset of Γ.

Definition

Let Γ be a group with the identity 1. If S is a subset of Γ satisfying

1. $1 \notin S$
2. if $s \in S$, then $s^{-1} \in S$.

then S is called a Cayley subset of Γ.

Example:

- Let $\Gamma = \mathbb{Z}_9$. Then $S = \{1, 3, 6, 8\}$ is a Cayley subset of Γ.
- If $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $S = \{(0, 1), (1, 0)\}$ is a Cayley subset of Γ.

Definition

Let S be a Cayley subset of a group Γ. The Cayley graph $Cay(\Gamma, S)$ is the graph with vertex set Γ, where $a, b \in E$ if $ab^{-1} \in S$.

Cayley graph

Definition

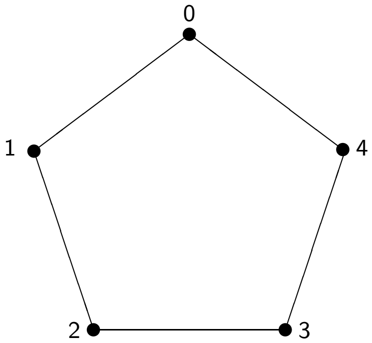
Let S be a Cayley subset of a group Γ. The Cayley graph $Cay(\Gamma, S)$ is the graph with vertex set Γ, where $a, b \in E$ if $ab^{-1} \in S$.

Remark:

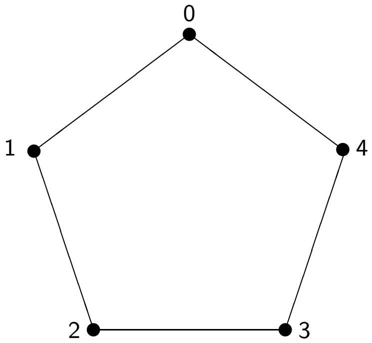
- The Cayley graph $Cay(\Gamma, S)$ is the undirected graph with vertex set Γ and edge set containing an edge from g to gs and from g to gs^{-1} whenever $g \in G$ and $s \in S$.

The Cayley graph $Cay(\mathbb{Z}_5, \{1, 4\})$:

The Cayley graph $Cay(\mathbb{Z}_5, \{1, 4\})$:



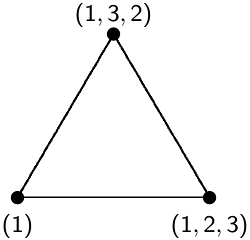
The Cayley graph $Cay(\mathbb{Z}_5, \{1, 4\})$:



Remark: Generally, $Cay(\mathbb{Z}_n, \{1, -1\})$ is the cycle C_n.

Let $\Gamma = S_3 = \{(1),(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}$ and $S = \{(1,2,3),(1,2,3)^{-1}\} = \{(1,2,3),(1,3,2)\}$.

Let $\Gamma = S_3 = \{(1),(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}$ and $S = \{(1,2,3),(1,2,3)^{-1}\} = \{(1,2,3),(1,3,2)\}$. Then $Cay(S_3, \{(1,2,3),(1,3,2)\})$ is given below:



- If $|S| = r$, then $Cay(\Gamma, S)$ is r-regular.
- $Cay(\Gamma, S)$ is connected if and only if S generates Γ.
- $Cay(\Gamma, S)$ is vertex-transitive.

- (Potočnik, Škoviera, Škrekovski, 2005) Every Cayley graph of degree at least 4 on an Abelian group admits a nowhere-zero 3-flow.

- (Potočnik, Škoviera, Škrekovski, 2005) Every Cayley graph of degree at least 4 on an Abelian group admits a nowhere-zero 3-flow.

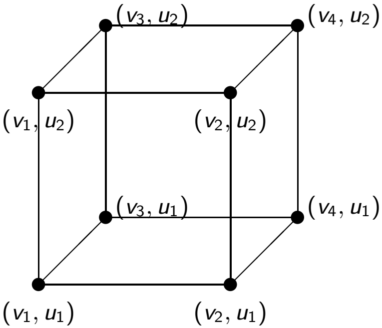
Motivated by Jeager's Conjecture, we prove that

- (Lai, Zhan, Zhang, Zhou) Every connected Cayley graph of degree at least 5 on an Abelian group is Z_3-connected.

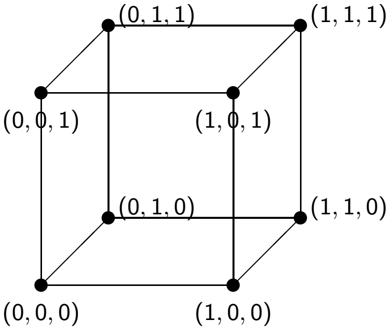
Definition: The Cartesian product $G_1 \Box G_2$ of two graphs G_1 and G_2 is defined on the vertex set $V(G_1) \times V(G_2)$, where two vertices (x_1, y_1) and (x_2, y_2) of $G_1 \Box G_2$ are adjacent if $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$, or $x_1 x_2 \in E(G_1)$ and $y_1 = y_2$. By definition, this operation of taking Cartesian products is commutative and associative.

Definition: The Cartesian product $G_1 \Box G_2$ of two graphs G_1 and G_2 is defined on the vertex set $V(G_1) \times V(G_2)$, where two vertices (x_1, y_1) and (x_2, y_2) of $G_1 \Box G_2$ are adjacent if $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$, or $x_1 x_2 \in E(G_1)$ and $y_1 = y_2$. By definition, this operation of taking Cartesian products is commutative and associative.

Example: $C_4 \times P_2$ is given below.



Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S = \{(1,0,0),(0,1,0),(0,0,1)\}$. Then $Cay(\Gamma, S)$ is is given below:

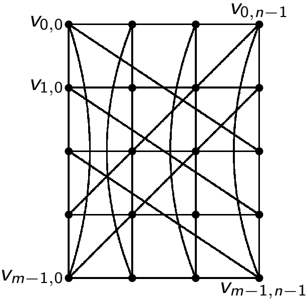


So, $Cay(\Gamma, S) = C_4 \times P_2$.

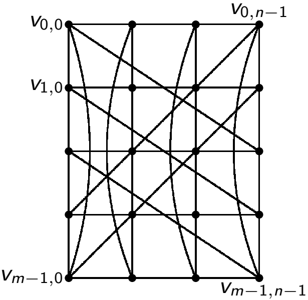
√ For any integer $m \geq 2$, the graphs $K_4 \Box P_m$ and $K_4 \Box C_m$ are Z_3-connected.

- √ For any integer $m \geq 2$, the graphs $K_4 \Box P_m$ and $K_4 \Box C_m$ are Z_3-connected.
- √ Let $m, n, t \geq 2$ be integers. Then the graphs $C_m \Box C_n \Box C_t$ and $C_m \Box C_n \Box P_t$ are Z_3-connected.

√ Assume $V(C_m) = \{x_0, x_1, \cdots, x_{m-1}\}(m \geq 2)$ and $E(C_m) = \{x_i x_{i+1} | i \in Z_m\}$, $V(P_n) = \{y_0, y_1, \cdots, y_{n-1}\}(n \geq 2)$ and $E(P_n) = \{y_i y_{i+1} | 0 \leq i \leq n-2\}$. For $i \in \{0, 1, \cdots, m-1\}$ and $j \in \{0, 1, \cdots, n-1\}$, denote v_{ij} the vertex (x_i, y_j) in $C_m \Box P_n$. For $0 \leq j \leq m-1$, let $B_j(m, n) = C_m \Box P_n + \bigcup_{i=0}^{m-1} \{v_{i,0} v_{i+j,n-1}\}$. Then $B_0(m, n) = C_m \Box C_n$.



√ Assume $V(C_m) = \{x_0, x_1, \cdots, x_{m-1}\}(m \geq 2)$ and $E(C_m) = \{x_i x_{i+1} | i \in Z_m\}$, $V(P_n) = \{y_0, y_1, \cdots, y_{n-1}\}(n \geq 2)$ and $E(P_n) = \{y_i y_{i+1} | 0 \leq i \leq n-2\}$. For $i \in \{0, 1, \cdots, m-1\}$ and $j \in \{0, 1, \cdots, n-1\}$, denote v_{ij} the vertex (x_i, y_j) in $C_m \Box P_n$. For $0 \leq j \leq m-1$, let $B_j(m,n) = C_m \Box P_n + \bigcup_{i=0}^{m-1} \{v_{i,0} v_{i+j,n-1}\}$. Then $B_0(m,n) = C_m \Box C_n$.



- Let $m, n, k \geq 2$. Then, for $0 \leq j \leq m-1$, $B_j(m,n) \Box P_k$ and $B_j(m,n) \Box C_k$ are Z_3-connected.

√ For an integer $m \geq 3$ and for $n \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Observe that it can be constructed from the m-cycle $x_0 x_1 x_2 \cdots x_{m-1} x_0$ by connecting x_i to x_{i+n} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

√ For an integer $m \geq 3$ and for $n \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Observe that it can be constructed from the m-cycle $x_0x_1x_2 \cdots x_{m-1}x_0$ by connecting x_i to x_{i+n} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

- Let s, m, and n be integers such that $m > n \geq 1$, $m \geq 3, s \geq 2$. If m is a multiple of n, then the graphs $C(m, n)\Box P_s$ and $C(m, n)\Box C_s$ are Z_3-connected.

- For an integer $m \geq 3$ and for $n \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Observe that it can be constructed from the m-cycle $x_0 x_1 x_2 \cdots x_{m-1} x_0$ by connecting x_i to x_{i+n} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

  - Let s, m, and n be integers such that $m > n \geq 1$, $m \geq 3, s \geq 2$. If m is a multiple of n, then the graphs $C(m, n) \Box P_s$ and $C(m, n) \Box C_s$ are Z_3-connected.

- For an integer $m \geq 3$ and for $r, s \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -r, r, -s, s\})$ will be denoted by $C(m, r, s)$. Observe that it can be constructed from the m-cycle $x_0 x_1 x_2 \cdots x_{m-1} x_0$ by connecting x_i to x_{i+r} and x_{i+s} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

√ For an integer $m \geq 3$ and for $n \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Observe that it can be constructed from the m-cycle $x_0 x_1 x_2 \cdots x_{m-1} x_0$ by connecting x_i to x_{i+n} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

- Let s, m, and n be integers such that $m > n \geq 1$, $m \geq 3, s \geq 2$. If m is a multiple of n, then the graphs $C(m, n) \square P_s$ and $C(m, n) \square C_s$ are Z_3-connected.

√ For an integer $m \geq 3$ and for $r, s \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -r, r, -s, s\})$ will be denoted by $C(m, r, s)$. Observe that it can be constructed from the m-cycle $x_0 x_1 x_2 \cdots x_{m-1} x_0$ by connecting x_i to x_{i+r} and x_{i+s} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

- Let s, r, and m be integers such that $m \geq 3$, $m > s \geq 1$ and $m > r \geq 1$. If $r | m$ and $s | m$, then $C(m, r, s)$ is Z_3-connected.

√ For an integer $m \geq 3$ and for $n \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Observe that it can be constructed from the m-cycle $x_0 x_1 x_2 \cdots x_{m-1} x_0$ by connecting x_i to x_{i+n} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

- Let s, m, and n be integers such that $m > n \geq 1$, $m \geq 3, s \geq 2$. If m is a multiple of n, then the graphs $C(m, n) \Box P_s$ and $C(m, n) \Box C_s$ are Z_3-connected.

√ For an integer $m \geq 3$ and for $r, s \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -r, r, -s, s\})$ will be denoted by $C(m, r, s)$. Observe that it can be constructed from the m-cycle $x_0 x_1 x_2 \cdots x_{m-1} x_0$ by connecting x_i to x_{i+r} and x_{i+s} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

- Let s, r, and m be integers such that $m \geq 3$, $m > s \geq 1$ and $m > r \geq 1$. If $r | m$ and $s | m$, then $C(m, r, s)$ is Z_3-connected.

√ Let $n \geq 2$ be an even integer. The Cayley graph $Cay(Z_n, \{-1, 1, \frac{n}{2}\})$ will be denoted by $C(n)$. Observe that it can be constructed from the n-cycle $x_0 x_1 \cdots x_n$ by connecting x_i to $x_{i+\frac{n}{2}}$ for each $i \in \{0, 1, \cdots, \frac{n}{2} - 1\}$.

√ For an integer $m \geq 3$ and for $n \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Observe that it can be constructed from the m-cycle $x_0x_1x_2 \cdots x_{m-1}x_0$ by connecting x_i to x_{i+n} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

- Let s, m, and n be integers such that $m > n \geq 1$, $m \geq 3$, $s \geq 2$. If m is a multiple of n, then the graphs $C(m, n)\Box P_s$ and $C(m, n)\Box C_s$ are Z_3-connected.

√ For an integer $m \geq 3$ and for $r, s \in Z_m - \{0\}$, the Cayley graph $Cay(Z_m, \{-1, 1, -r, r, -s, s\})$ will be denoted by $C(m, r, s)$. Observe that it can be constructed from the m-cycle $x_0x_1x_2 \cdots x_{m-1}x_0$ by connecting x_i to x_{i+r} and x_{i+s} (indices taken modulo m) for each $i \in \{0, 1, \cdots, m-1\}$.

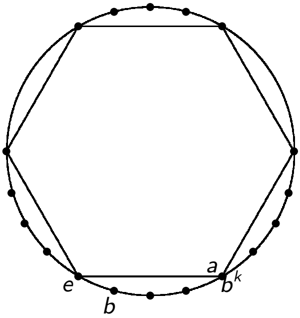
- Let s, r, and m be integers such that $m \geq 3$, $m > s \geq 1$ and $m > r \geq 1$. If $r|m$ and $s|m$, then $C(m, r, s)$ is Z_3-connected.

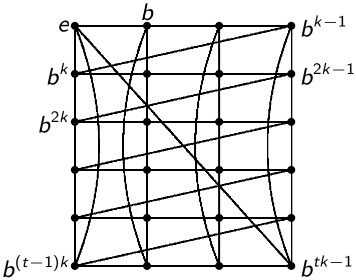
√ Let $n \geq 2$ be an even integer. The Cayley graph $Cay(Z_n, \{-1, 1, \frac{n}{2}\})$ will be denoted by $C(n)$. Observe that it can be constructed from the n-cycle $x_0x_1 \cdots x_n$ by connecting x_i to $x_{i+\frac{n}{2}}$ for each $i \in \{0, 1, \cdots, \frac{n}{2} - 1\}$.

- $C(2m)\Box C_n$ is Z_3-connected.

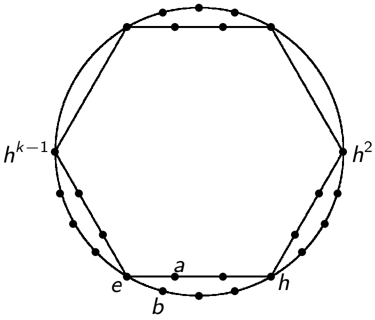
(1) Let $G = Cay(\Gamma, S)$ be a Cayley graph and $a, b \in S$ with $o(a) = t \geq 3$ and $o(b) = s \geq 3$. If $\langle a \rangle \cap \langle b \rangle = \{e\}$, then $\langle a, b \rangle = \{a^i b^j | 0 \leq i \leq t-1, 0 \leq j \leq s-1\}$ and the subgraph induced by the a-edges and b-edges is $C_t \Box C_s$.

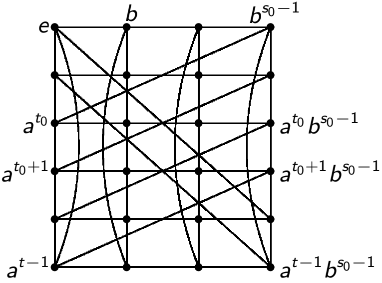
(2) Let $G = Cay(\Gamma, S)$ be a Cayley graph and $a, b \in S$ with $o(a) = t \geq 3$ and $o(b) = s \geq 3$. If $\langle a \rangle \subseteq \langle b \rangle$, then $t|s$. Assume that $s = tk$. Then the subgraph induced by the a-edges and b-edges is $C(s, k) = B_1(t, k)$.





(3) Let $G = Cay(\Gamma, S)$ be a Cayley graph and $a, b \in S$ with $o(a) = t \geq 3$ and $o(b) = s \geq 3$. If $\langle h \rangle = \langle a \rangle \cap \langle b \rangle$ is a non-trivial subgroup of $\langle a \rangle$ and $\langle b \rangle$, then there are two smallest positive integers $s_0, t_0 \geq 2$ such that $h = a^{t_0} = b^{s_0}$. Thus $\langle a, b \rangle = \{a^i b^j | 0 \leq i \leq t-1, 0 \leq j \leq s_0 - 1\} = \{a^i b^j | 0 \leq i \leq t_0 - 1, 0 \leq j \leq s - 1\}$, and the subgraph induced by the a-edges and b-edges is $B_{t_0}(t, s_0) = B_{s_0}(s, t_0)$, where $k = o(h)$, and $t = kt_0$ and $s = ks_0$.





(4) Every connected Cayley graph of an Abelian group and of degree 5 is Z_3-connected.

(4) Every connected Cayley graph of an Abelian group and of degree 5 is Z_3-connected.

Proof. Let Γ be an Abelian group with identity $e \in \Gamma$, and let $G = Cay(\Gamma; S)$ be a connected Cayley graph of degree 5 on Γ. Then S is a generator set of Γ.

(4) Every connected Cayley graph of an Abelian group and of degree 5 is Z_3-connected.

Proof. Let Γ be an Abelian group with identity $e \in \Gamma$, and let $G = Cay(\Gamma; S)$ be a connected Cayley graph of degree 5 on Γ. Then S is a generator set of Γ. Since $|S| = 5$, we have either $S = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, or $S = \{\tau_1, \tau_2, \tau_3, a, a^{-1}\}$, or $S = \{\tau_1, b, b^{-1}, c, c^{-1}\}$, where τ_i is an involution, and the orders of a, b, c are at least 3.

(4) Every connected Cayley graph of an Abelian group and of degree 5 is Z_3-connected.

Proof. Let Γ be an Abelian group with identity $e \in \Gamma$, and let $G = Cay(\Gamma; S)$ be a connected Cayley graph of degree 5 on Γ. Then S is a generator set of Γ. Since $|S| = 5$, we have either $S = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, or $S = \{\tau_1, \tau_2, \tau_3, a, a^{-1}\}$, or $S = \{\tau_1, b, b^{-1}, c, c^{-1}\}$, where τ_i is an involution, and the orders of a, b, c are at least 3.

- $S = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$.
- $S = \{\tau_1, \tau_2, \tau_3, a, a^{-1}\}$.
- $S = \{\tau_1, b, b^{-1}, c, c^{-1}\}$.

Outline of the Proof

(5) Every connected Cayley graph of an Abelian group and of degree at least 5 is Z_3-connected.

(5) Every connected Cayley graph of an Abelian group and of degree at least 5 is Z_3-connected.

Proof.

- Let Γ be an Abelian group with identity $e \in \Gamma$, and let $G = Cay(\Gamma; S)$ be a connected Cayley graph of degree at least 5 on Γ. Then $|S| \geq 5$ and S is a generator set of Γ.

(5) Every connected Cayley graph of an Abelian group and of degree at least 5 is Z_3-connected.

Proof.

- Let Γ be an Abelian group with identity $e \in \Gamma$, and let $G = Cay(\Gamma; S)$ be a connected Cayley graph of degree at least 5 on Γ. Then $|S| \geq 5$ and S is a generator set of Γ.
- Let F be the set of edges of the Cayley graph G induced by two involutions or by a generator of order ≥ 3. Then F is a 2-factor of G. If $G - F$ is Z_3-connected, then so is G. Therefore, it is sufficient to prove the theorem for $|S| = 6$.

(5) Every connected Cayley graph of an Abelian group and of degree at least 5 is Z_3-connected.

Proof.

- Let Γ be an Abelian group with identity $e \in \Gamma$, and let $G = Cay(\Gamma; S)$ be a connected Cayley graph of degree at least 5 on Γ. Then $|S| \geq 5$ and S is a generator set of Γ.
- Let F be the set of edges of the Cayley graph G induced by two involutions or by a generator of order ≥ 3. Then F is a 2-factor of G. If $G - F$ is Z_3-connected, then so is G. Therefore, it is sufficient to prove the theorem for $|S| = 6$.
- Let F' be the set of edges of Γ induced by one involution. Then F' is a 1-factor of G. So, it suffices to prove theorem when $S = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$, where the orders of a, b, c are at least three. Let the orders of a, b and c are t, s, and r, respectively.

(5) Every connected Cayley graph of an Abelian group and of degree at least 5 is Z_3-connected.

Proof.

- Let Γ be an Abelian group with identity $e \in \Gamma$, and let $G = Cay(\Gamma; S)$ be a connected Cayley graph of degree at least 5 on Γ. Then $|S| \geq 5$ and S is a generator set of Γ.

- Let F be the set of edges of the Cayley graph G induced by two involutions or by a generator of order ≥ 3. Then F is a 2-factor of G. If $G - F$ is Z_3-connected, then so is G. Therefore, it is sufficient to prove the theorem for $|S| = 6$.

- Let F' be the set of edges of Γ induced by one involution. Then F' is a 1-factor of G. So, it suffices to prove theorem when $S = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$, where the orders of a, b, c are at least three. Let the orders of a, b and c are t, s, and r, respectively.

Case 1. $\langle a \rangle \cap (\langle b \rangle \cup \langle c \rangle) = \{e\}$.

Case 2. $\langle a \rangle \cap (\langle b \rangle \cup \langle c \rangle) \neq \{e\}$, $\langle b \rangle \cap (\langle a \rangle \cup \langle c \rangle) \neq \{e\}$, and $\langle c \rangle \cap (\langle a \rangle \cup \langle b \rangle) \neq \{e\}$.

Thank You!