Recent Research Topics in Graph Theory

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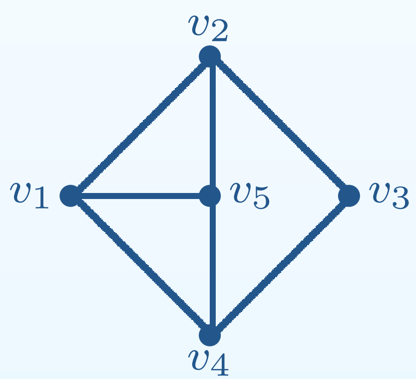
Fall 2008

Section 1

An Overview on Graph Theory

Definition of a Graph

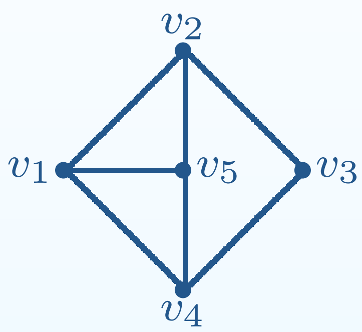
A graph is a pair $G = (V, E)$ of sets satisfying $E \subseteq V \times V$. The elements of V are the vertices of the graph G, the elements of E are its edges.



G

Degree

√ The neighborhood $N(v)$ of a vertex v: the set of all vertices that are adjacent to v.



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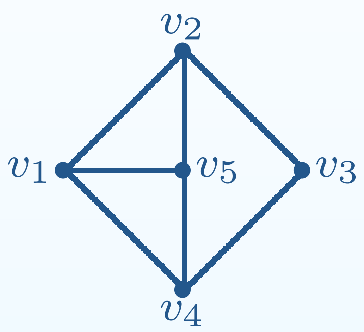

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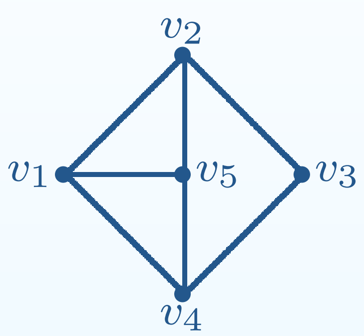


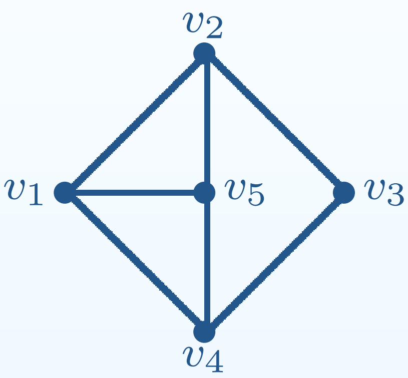
- √ The degree $d(v)$ of a vertex v is the number of vertices in $N(v)$.

- √ The minimum degree of the graph is

$$\delta(G) = \min\left\{d(v) | v \in V(G)\right\}$$

Connectivity

√ The graph G is connected if any two of its vertices are linked by a path in G.



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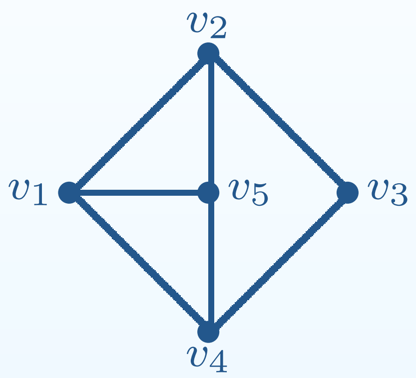

- √ G is called k-connected if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.

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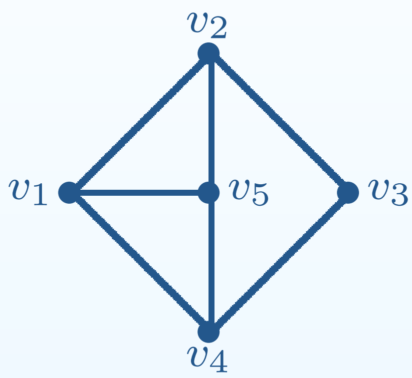

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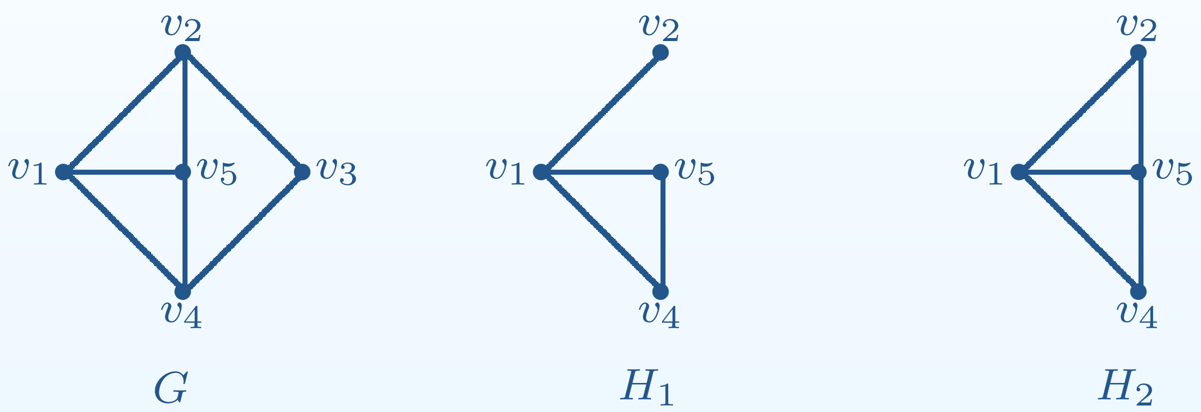
- The connectivity of G is the minimum k for which G is k-connected.

Subgraphs

√ Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G, written as $G' \subseteq G$.



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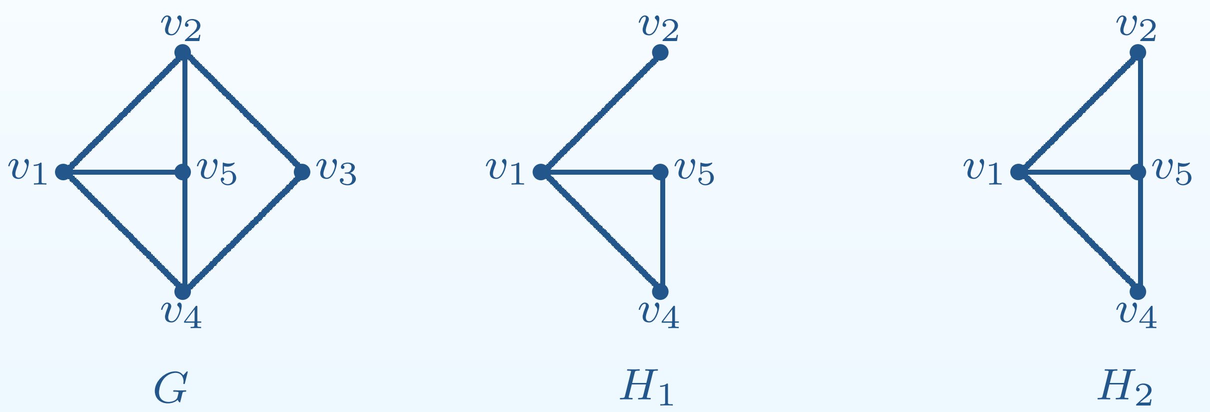
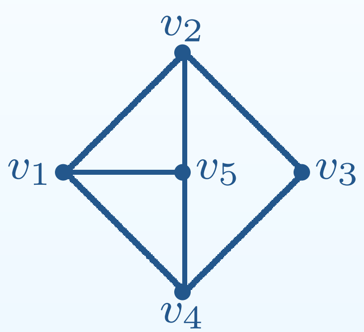


- If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph* of G, written as $G' = G[V']$.

Hamiltonian Graphs

√ G is *hamiltonian* if G has a cycle containing all vertices of G.



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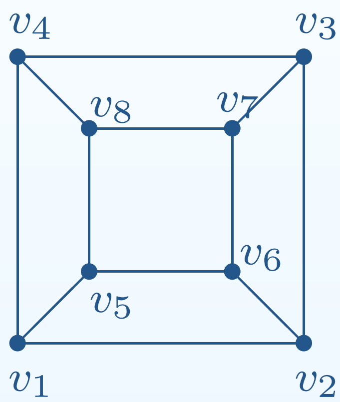

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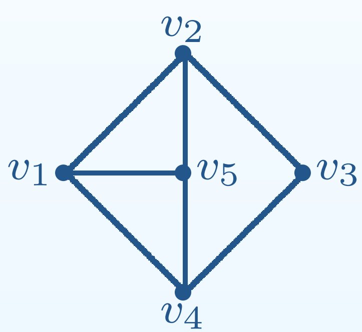
Section 2

Recent Research I: Claw-free Graphs

Claw-free Graphs

√ The four-vertex star $K_{1,3}$ is called a claw. The degree 3 vertex in a claw is called the center of a claw.



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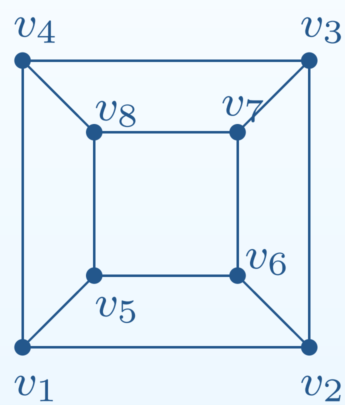

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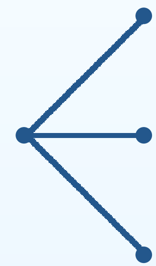

$K_{1,3}$

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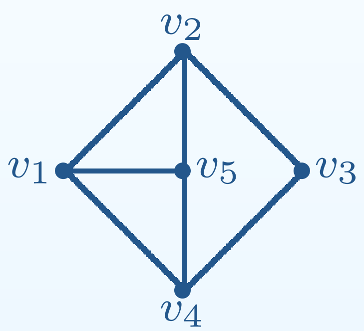

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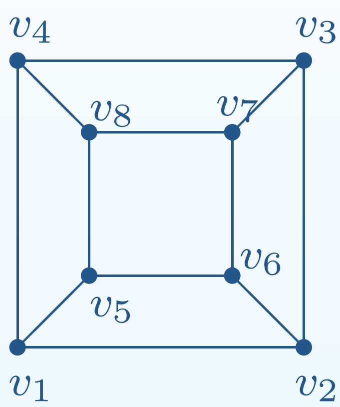

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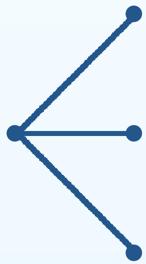

$K_{1,3}$

- √ If G contains no copy of $K_{1,3}$ as an induced subgraph, then G is *claw-free.*

Conjecture

√ **Conjecture(Matthews and Sumner, 1986):** Every 4-connected claw-free graph is hamiltonian.

Problems

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$$\Big\{\text{claw-free graphs}\Big\} \subseteq \Big\{\text{quasi claw-free graphs}\Big\}$$

$$\Big\{\text{claw-free graphs}\Big\} \subseteq \Big\{\text{almost claw-free graphs}\Big\}$$

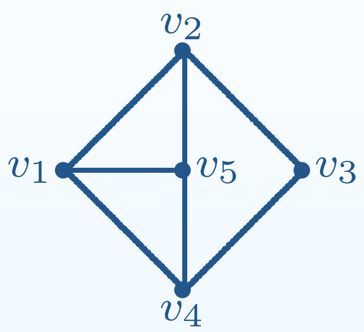
Can we generalize the results for claw-free graphs to either quasi claw-free graphs or almost claw-free graphs?

Section 3

Recent Research II: Supereulerian Graphs

Supereulerian Graphs

√ The degree $d(v)$ of a vertex v is the number of vertices that are adjacent to v.



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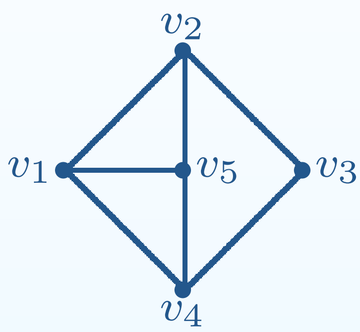

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Supereulerian Graphs

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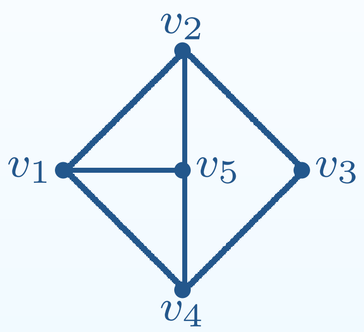
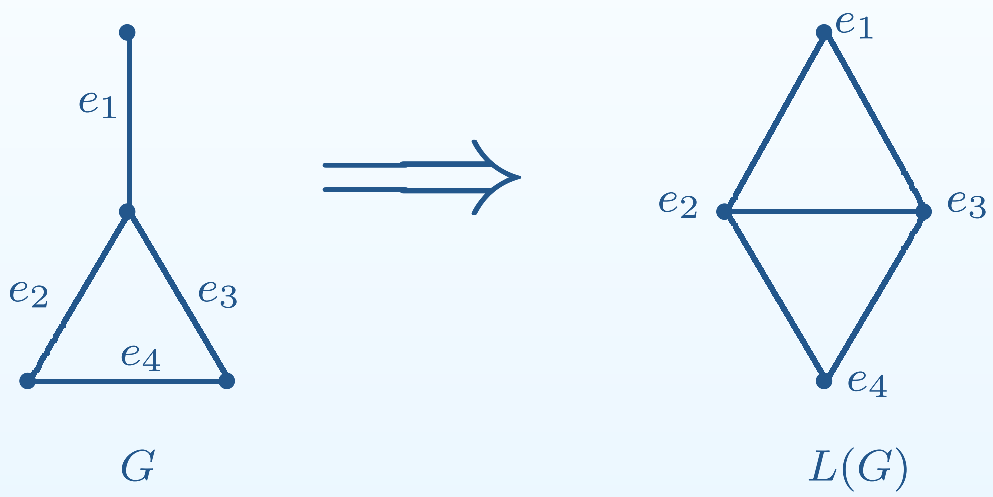


- G is called an even graph if the degree of each vertex is even.
- G is called supereulerian if G has an even subgraph.

Line Graph

√ The *line graph* of a graph G, denote by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.



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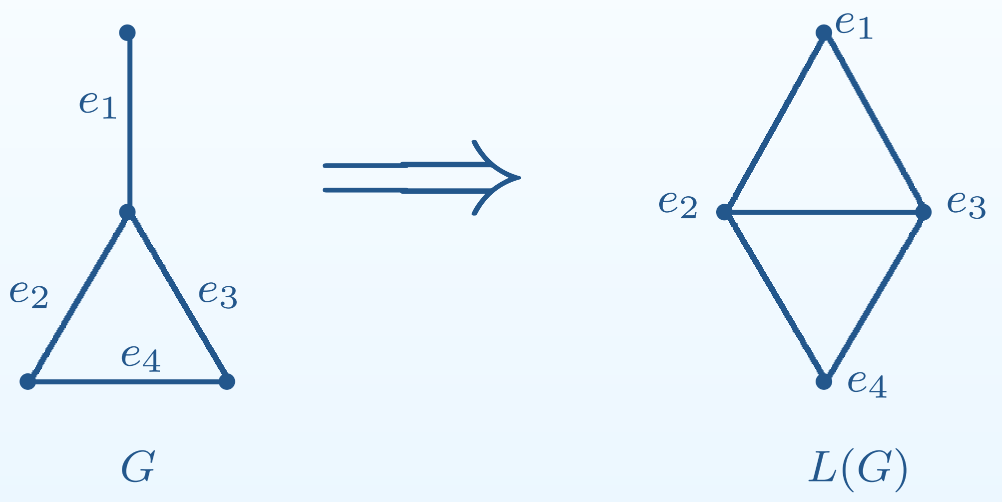

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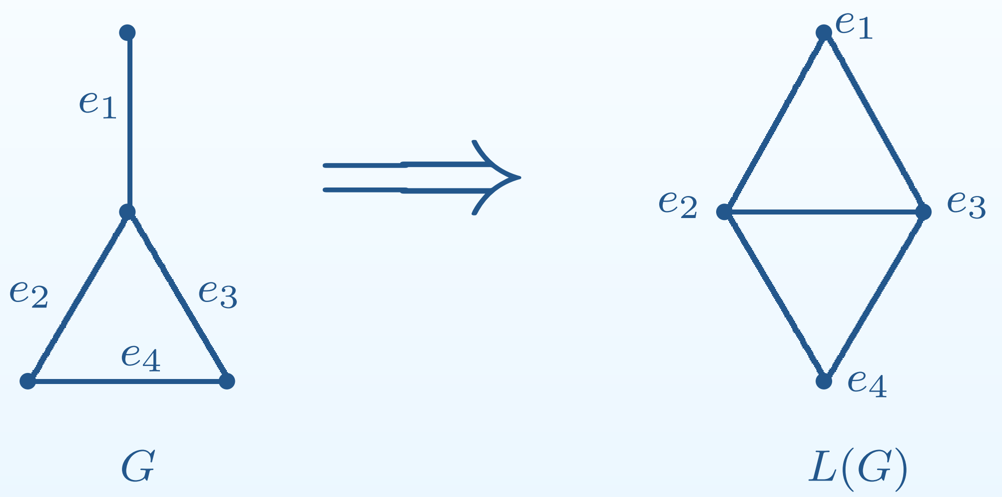

- The line graph is claw-free.

- G is k-connected if and only if $L(G)$ is essentially k-edge-connected.

Conjecture

√ **Conjecture(Thomassen, 1985):** Every 4-connected line graph is hamiltonian.

Harary and Nash-Williams Theorem

Theorem (Harary and Nash-Williams): Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a connected dominating even subgraph.

$$H = L(G)$$

H .... G

(Hamiltonian) (DES)

Conjectures and Projects

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- √ **Project 3:** 3-edge-connected + essentially 6-edge-connected + other condition $\Longrightarrow$ supereulerian.

Section 4

Recent Research III: Vertex-disjoint Subgraphs

Background

- √ Suppose that several first-aid workers from around the world have volunteered for a rescue mission in some country that has been struck by a disaster. The volunteers are to be divided into two-person teams, but members of each team must speak the same language.

  (a) What is the maximum number of pairs that can be sent out on a rescue mission?

  (b) What if each team has 3 people?

  (c) If we use vertices to represent all volunteers, and use an edge to join two vertices if two people can speak the same language, then the problem is actually to find the maximum number of vertex-disjoint subgraphs.

Research Work - 01

- √ So far, no one has attempted to solve the number of vertex-disjoint subgraphs with 6 vertices, since finding the number of vertex-disjoint subgraphs with 5 vertices is very difficult already.

- √ What condition can guarantee the existence of vertex-disjoint subgraphs with 6 vertices?

Research Work - 02

- √ The directed graph is a model for a roadmap. The vertices represent landmarks and the directed edges represent the one-way streets.
- √ Currently, there are almost no publications on the number of vertex-disjoint subgraphs in a directed graph.
- √ What condition can guarantee the existence of vertex-disjoint subgraphs with 3 or 4 vertices in a directed graph?

Research Work - 03

- √ In a school, there are m teachers and n classes. Each teacher teaches classes during some periods. To schedule a complete timetable, we can model this as a problem in graph theory: The vertices represent teachers and classes, and if a teacher teaches a class, then we use an edge to join these two vertices. Therefore, there are no edges joining two teachers, or two classes.

- √ For a graph, if the vertices can be divided into to groups, say X and Y, and any two vertices in X or Y cannot be joined by an edge, then this graph is called a bipartite graph.

- √ What condition can guarantee the existence of vertex-disjoint subgraphs in a graph?

Thank you!