

Chapter 4

The Tangent Space

In this chapter we study the vector space tangent to the trace of a regular patch at a particular point.

4.1 Tangent Vectors and Directional Derivatives

In order to distinguish between vectors with same magnitude and direction but different initial points, we make the following definition:

Definition 61 *Let $\mathbf{p} \in \mathbb{R}^n$. A tangent vector to \mathbb{R}^n at \mathbf{p} , denoted by $\mathbf{v}_{\mathbf{p}}$, is an ordered pair (\mathbf{v}, \mathbf{p}) where $\mathbf{v} \in \mathbb{R}^n$. The vector \mathbf{v} is called the vector part; the point \mathbf{p} is called the point of application of $\mathbf{v}_{\mathbf{p}}$. Two tangent vectors $\mathbf{v}_{\mathbf{p}}$ and $\mathbf{w}_{\mathbf{q}}$ are equal if and only if $\mathbf{v} = \mathbf{w}$ and $\mathbf{p} = \mathbf{q}$.*

Note that $\mathbf{v}_{\mathbf{p}}$ can be thought of as an arrow from point \mathbf{p} to the point $\mathbf{p} + \mathbf{v}$; however, we are not free to translate $\mathbf{v}_{\mathbf{p}}$ to some other arbitrary position. Since $\mathbf{v}_{\mathbf{p}}$ is as a $2n$ -tuple, another way to think of $\mathbf{v}_{\mathbf{p}}$ is as a vector in \mathbb{R}^{2n} . Now let $\mathbb{R}_{\mathbf{p}}^n = \{\mathbf{v}_{\mathbf{p}} \mid \mathbf{v} \in \mathbb{R}^n\}$ denote the set of all tangent vectors at \mathbf{p} and define addition and scalar multiplication in $\mathbb{R}_{\mathbf{p}}^n$ as follows:

$$\mathbf{v}_{\mathbf{p}} + \mathbf{w}_{\mathbf{p}} = (\mathbf{v} + \mathbf{w})_{\mathbf{p}}$$

and

$$\lambda \mathbf{v}_{\mathbf{p}} = (\lambda \mathbf{v})_{\mathbf{p}}.$$

Thus we add and scalar multiply in the first n components only; the last n coordinates are always the components of \mathbf{p} . Then $\mathbb{R}_{\mathbf{p}}^n$ with this addition and scalar multiplication is a vector space isomorphic to \mathbb{R}^n .

Definition 62 Let $\mathbf{p} \in \mathbb{R}^n$. The tangent space of \mathbb{R}^n at \mathbf{p} is the n -dimensional vector space

$$\mathbb{R}_{\mathbf{p}}^n = \{\mathbf{v}_{\mathbf{p}} \mid \mathbf{v} \in \mathbb{R}^n\}.$$

Define the Euclidean inner product in $\mathbb{R}_{\mathbf{p}}^n$ by

$$\mathbf{v}_{\mathbf{p}} \bullet \mathbf{w}_{\mathbf{p}} = \mathbf{v} \bullet \mathbf{w}.$$

When $n = 2$ define the complex structure J on $\mathbb{R}_{\mathbf{p}}^2$ by

$$J(\mathbf{v}_{\mathbf{p}}) = [J(\mathbf{v})]_{\mathbf{p}}.$$

And when $n = 3$ define the cross product on $\mathbb{R}_{\mathbf{p}}^3$ by

$$\mathbf{v}_{\mathbf{p}} \times \mathbf{w}_{\mathbf{p}} = (\mathbf{v} \times \mathbf{w})_{\mathbf{p}}.$$

Then all facts and identities that hold for the Euclidean inner product and the cross-product hold in the tangent space at \mathbf{p} .

Definition 63 Let $\mathbf{v} \in \mathbb{R}^n$ be a non-zero vector, let $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued differentiable function. The directional derivative of f in the direction of \mathbf{v} at point $\mathbf{p} \in \mathbb{R}^n$, denoted by $\mathcal{D}_{\mathbf{v}}(f)(\mathbf{p})$, measures the rate of change of f along the line $\mathbf{p} + t\mathbf{u}$ at the instant $t = 0$ and is defined by

$$\mathcal{D}_{\mathbf{v}}(f)(\mathbf{p}) = \lim_{t \rightarrow 0} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{t} = \left. \frac{d}{dt} [f(\mathbf{p} + t\mathbf{u})] \right|_{t=0}.$$

Thus, given a tangent vector $\mathbf{v}_{\mathbf{p}}$, the directional derivative $\mathcal{D}_{\mathbf{v}}(-)(\mathbf{p})$ is a function that assigns a real number to each real-valued differentiable function. We shall think of a tangent vector $\mathbf{v}_{\mathbf{p}}$ as a kind of directional derivative.

Definition 64 Let $\mathbf{v}_{\mathbf{p}} \in \mathbb{R}_{\mathbf{p}}^n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable. Then $\mathbf{v}_{\mathbf{p}}$ acts on f in the following way:

$$\mathbf{v}_{\mathbf{p}} [f] = \lim_{t \rightarrow 0} \frac{f(\mathbf{p} + t\mathbf{v}) - f(\mathbf{p})}{t} = \left. \frac{d}{dt} [f(\mathbf{p} + t\mathbf{v})] \right|_{t=0}.$$

Note that $\mathbf{v}_{\mathbf{p}}[f]$ differs from $\mathcal{D}_{\mathbf{v}}(f)(\mathbf{p})$ to the extent that $\mathcal{D}_{\mathbf{v}}(f)(\mathbf{p})$ depends only on the direction of \mathbf{v} and not its magnitude. This important difference is clarified by the next lemma, which is important in computations:

Lemma 65 *Let $\mathbf{v}_{\mathbf{p}} \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable. Then*

$$\mathbf{v}_{\mathbf{p}}[f] = \nabla f(\mathbf{p}) \bullet \mathbf{v}.$$

Proof. By the chain rule,

$$\frac{d}{dt}[f(\mathbf{p}+t\mathbf{v})] = \nabla f(\mathbf{p}+t\mathbf{v}) \bullet \frac{d}{dt}(\mathbf{p}+t\mathbf{v}) = \nabla f(\mathbf{p}+t\mathbf{v}) \bullet \mathbf{v}.$$

Evaluating at $t = 0$ gives the result. ■

If $t = \|\mathbf{v}\|$, then $\mathbf{v} = t\mathbf{u}$ and

$$\mathbf{v}_{\mathbf{p}}[f] = \nabla f(\mathbf{p}) \bullet \mathbf{v} = \nabla f(\mathbf{p}) \bullet t\mathbf{u} = t\mathcal{D}_{\mathbf{u}}(f)(\mathbf{p}).$$

Exercise 4.1.1 *Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable. If $\mathbf{v}_{\mathbf{p}} \in \mathbb{R}_{\mathbf{p}}^n$ and $\lambda, \rho \in \mathbb{R}$, show that:*

1. $\mathbf{v}_{\mathbf{p}}[\lambda f + \rho g] = \lambda \mathbf{v}_{\mathbf{p}}[f] + \rho \mathbf{v}_{\mathbf{p}}[g]$
2. $\mathbf{v}_{\mathbf{p}}[fg] = \mathbf{v}_{\mathbf{p}}[f]g(\mathbf{p}) + f(\mathbf{p})\mathbf{v}_{\mathbf{p}}[g]$
3. $\mathbf{v}_{\mathbf{p}}[1/g] = -\mathbf{v}_{\mathbf{p}}[g]/[g(\mathbf{p})]^2$
4. $\mathbf{v}_{\mathbf{p}}[f/g] = \{\mathbf{v}_{\mathbf{p}}[f]g(\mathbf{p}) - f(\mathbf{p})\mathbf{v}_{\mathbf{p}}[g]\}/[g(\mathbf{p})]^2$.

4.2 Tangent Maps

In this section we analyze the precise relationship between a tangent vector at a point \mathbf{p} in \mathbb{R}^2 and an important related tangent vector at the point $\mathbf{q} = \mathbf{x}(\mathbf{p})$ on the trace of a patch $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. This relationship is critical to understanding the local curvature of a patch. Although we work almost exclusively with patches $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, much of this discussion generalizes directly to differentiable maps $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Definition 66 Let \mathcal{U} be an open subset of \mathbb{R}^2 , let $\mathbf{p} \in \mathcal{U}$. Let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a patch and let $\mathbf{q} = \mathbf{x}(\mathbf{p})$. The tangent map of \mathbf{x} at \mathbf{p} is the map $\mathbf{x}_* : \mathbb{R}_{\mathbf{p}}^2 \rightarrow \mathbb{R}_{\mathbf{q}}^3$ defined by

$$\mathbf{x}_*(\mathbf{v}_{\mathbf{p}}) = \left(\left. \frac{d}{dt} [\mathbf{x}(\mathbf{p}+t\mathbf{v})] \right|_{t=0} \right)_{\mathbf{q}}.$$

The composition $\mathbf{x}(\mathbf{p}+t\mathbf{v})$ is a parametrized curve passing through \mathbf{q} at time $t = 0$; its derivative is the velocity at time t . So $\mathbf{x}_*(\mathbf{v}_{\mathbf{p}})$ is the tangent vector at \mathbf{q} whose vector part is the velocity of the curve $\mathbf{x}(\mathbf{p}+t\mathbf{v})$ at time $t = 0$. In particular, $\mathbf{x}(\mathbf{p}+t\mathbf{e}_1)$ is the first coordinate curve passing through \mathbf{q} at time $t = 0$. From our earlier calculations in Chapter 3 (see 3.1), the derivative at time $t = 0$ is $\mathbf{x}_u(\mathbf{p})$. Therefore

$$\mathbf{x}_* \left[(\mathbf{e}_1)_{\mathbf{p}} \right] = [\mathbf{x}_u(\mathbf{p})]_{\mathbf{q}}$$

and similarly

$$\mathbf{x}_* \left[(\mathbf{e}_2)_{\mathbf{p}} \right] = [\mathbf{x}_v(\mathbf{p})]_{\mathbf{q}}.$$

Theorem 67 Let \mathcal{U} be an open subset of \mathbb{R}^2 , let $\mathbf{p} \in \mathcal{U}$, and let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a patch with $\mathbf{q} = \mathbf{x}(\mathbf{p})$. If $\mathbf{x} = (f, g, h)$ then

$$\mathbf{x}_*(\mathbf{v}_{\mathbf{p}}) = (\mathbf{v}_{\mathbf{p}}[f], \mathbf{v}_{\mathbf{p}}[g], \mathbf{v}_{\mathbf{p}}[h])_{\mathbf{q}}.$$

Proof.

$$\begin{aligned} \mathbf{x}_*(\mathbf{v}_{\mathbf{p}}) &= \left(\left. \frac{d}{dt} [\mathbf{x}(\mathbf{p}+t\mathbf{v})] \right|_{t=0} \right)_{\mathbf{q}} = \left(\left. \frac{d}{dt} (f(\mathbf{p}+t\mathbf{v}), g(\mathbf{p}+t\mathbf{v}), h(\mathbf{p}+t\mathbf{v})) \right|_{t=0} \right)_{\mathbf{q}} \\ &= \left(\left. \frac{d}{dt} [f(\mathbf{p}+t\mathbf{v})] \right|_{t=0}, \left. \frac{d}{dt} [g(\mathbf{p}+t\mathbf{v})] \right|_{t=0}, \left. \frac{d}{dt} [h(\mathbf{p}+t\mathbf{v})] \right|_{t=0} \right)_{\mathbf{q}} \\ &= (\mathbf{v}_{\mathbf{p}}[f], \mathbf{v}_{\mathbf{p}}[g], \mathbf{v}_{\mathbf{p}}[h])_{\mathbf{q}}. \end{aligned}$$

■

Corollary 68 Let \mathcal{U} be an open subset of \mathbb{R}^2 , let $\mathbf{p} \in \mathcal{U}$, and let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a patch with $\mathbf{q} = \mathbf{x}(\mathbf{p})$. Then the tangent map $\mathbf{x}_* : \mathbb{R}_{\mathbf{p}}^2 \rightarrow \mathbb{R}_{\mathbf{q}}^3$ is linear.

Proof. Left to the reader. ■

Example 69 Define $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $\mathbf{x}(u, v) = (uv, u^2 + v^2, u^3 - v)$. Let us describe the tangent map at the point $\mathbf{p} = (2, 1)$. Let (x, y) be the vector part of a general tangent vector $\mathbf{v}_{\mathbf{p}}$; note that $\mathbf{q} = \mathbf{x}(\mathbf{p}) = (2, 5, 7)$. By Theorem 67

$$\begin{aligned} \mathbf{x}_* \left((x, y)_{\mathbf{p}} \right) &= \left(\mathbf{v}_{\mathbf{p}}[uv], \mathbf{v}_{\mathbf{p}}[u^2 + v^2], \mathbf{v}_{\mathbf{p}}[u^3 - v] \right)_{(2,5,7)} \\ &= \left(\nabla(uv)|_{(2,1)} \bullet \mathbf{v}, \nabla(u^2 + v^2)|_{(2,1)} \bullet \mathbf{v}, \nabla(u^3 - v)|_{(2,1)} \bullet \mathbf{v} \right)_{(2,5,7)} \\ &= \left((1, 2) \bullet \mathbf{v}, (4, 2) \bullet \mathbf{v}, (12, -1) \bullet \mathbf{v} \right)_{(2,5,7)} \\ &= (x + 2y, 4x + 2y, 12x - y)_{(2,5,7)}, \end{aligned}$$

which is a linear bijective map from the tangent space at $\mathbf{p} = (2, 1)$ to the tangent space at $\mathbf{q} = (2, 5, 7)$.

We have defined $\mathbf{x}_*(\mathbf{v}_{\mathbf{p}})$ to be the tangent vector whose vector part is $\frac{d}{dt} [\mathbf{x}(\mathbf{p} + t\mathbf{v})]|_{t=0}$ and whose point of application is $\mathbf{q} = \mathbf{x}(\mathbf{p})$. Given a tangent vector $\mathbf{v}_{\mathbf{p}}$, let $\beta(t) = \mathbf{p} + t\mathbf{v}$ and observe that $\beta(0) = \mathbf{p}$ and $\beta'(0) = \mathbf{v}$. Thus

$$\mathbf{x}_*(\mathbf{v}_{\mathbf{p}}) = [(\mathbf{x} \circ \beta)'(0)]_{\mathbf{q}}. \quad (4.1)$$

More generally, let $\varepsilon > 0$ and let $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ be any curve for which $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}$. The next theorem says that formula (4.1) holds for general curves α as well.

Theorem 70 Let \mathcal{U} be an open subset of \mathbb{R}^2 and let $\mathbf{p} \in \mathcal{U}$. Let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a patch and let $\mathbf{q} = \mathbf{x}(\mathbf{p})$. Let $\mathbf{v}_{\mathbf{p}} \in \mathbb{R}_{\mathbf{p}}^2$, let $\varepsilon > 0$, and let $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ be any curve such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}$. Then

$$\mathbf{x}_*(\mathbf{v}_{\mathbf{p}}) = [(\mathbf{x} \circ \alpha)'(0)]_{\mathbf{q}}.$$

Proof. *Proof:* Left to the reader. ■

The lesson to be learned here, is that the tangent map pays no attention to the curve “carrying” the tangent vector $\mathbf{v}_{\mathbf{p}}$. All that matters is that at the instant $t = 0$, the curve have velocity \mathbf{v} and position \mathbf{p} . So we are free to choose the curve most convenient for our immediate purposes.

If $L : V \rightarrow W$ is a non-injective linear map of vector spaces, then $L(\mathbf{v}) = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$. To see this, note that if $\mathbf{v}_1 \neq \mathbf{v}_2$, then $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \neq \mathbf{0}$. So if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ and $\mathbf{v}_1 \neq \mathbf{v}_2$, then $L(\mathbf{v}) = L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{0}$. We say that \mathbf{v} lies in the *kernel* of L , denoted by $\ker(L)$, if and only if $L(\mathbf{v}) = \mathbf{0}$.

Theorem 71 Let \mathcal{U} be an open subset of \mathbb{R}^2 and let $\mathbf{p} \in \mathcal{U}$. Let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a patch and let $\mathbf{q} = \mathbf{x}(\mathbf{p})$. Then the tangent map $\mathbf{x}_* : \mathbb{R}_{\mathbf{p}}^2 \rightarrow \mathbb{R}_{\mathbf{q}}^3$ is injective if and only if $\text{rank}[\mathcal{J}(\mathbf{x})(\mathbf{p})] = 2$.

Proof. Let $\mathbf{v}_{\mathbf{p}} \in \mathbb{R}_{\mathbf{p}}^2$ and let $\mathbf{x} = (f, g, h)$. If $\mathbf{v}_{\mathbf{p}} \in \ker(\mathbf{x}_*)$ then by Theorem 67 we have $\mathbf{v}_{\mathbf{p}}[f] = \mathbf{v}_{\mathbf{p}}[g] = \mathbf{v}_{\mathbf{p}}[h] = 0$. By Lemma 65, $\nabla f(\mathbf{p}) \bullet \mathbf{v} = \nabla g(\mathbf{p}) \bullet \mathbf{v} = \nabla h(\mathbf{p}) \bullet \mathbf{v} = 0$. Hence

$$[\mathbf{v}] \mathcal{J}(\mathbf{x})(\mathbf{p}) = [\mathbf{v}] [\nabla f(\mathbf{p}) \mid \nabla g(\mathbf{p}) \mid \nabla h(\mathbf{p})] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (4.2)$$

Now if \mathbf{x}_* is not injective, there exists some non-zero $\mathbf{v}_{\mathbf{p}} \in \ker(\mathbf{x}_*)$. Evaluating equation (4.2) at the vector part $\mathbf{v} = (a, b) \neq \mathbf{0}$ of this tangent vector gives

$$\mathbf{0} = [a \mid b] \begin{bmatrix} \mathbf{x}_u(\mathbf{p}) \\ \mathbf{x}_v(\mathbf{p}) \end{bmatrix} = a\mathbf{x}_u(\mathbf{p}) + b\mathbf{x}_v(\mathbf{p}),$$

which says that $\mathbf{x}_u(\mathbf{p})$ and $\mathbf{x}_v(\mathbf{p})$ are linearly dependent; equivalently, $\text{rank}(\mathcal{J}(\mathbf{x})(\mathbf{p})) < 2$ by Lemma 56. Conversely, if the $\text{rank}(\mathcal{J}(\mathbf{x})(\mathbf{p})) < 2$, one can reverse the argument above to obtain a non-zero tangent vector $\mathbf{v}_{\mathbf{p}} \in \ker(\mathbf{x}_*)$, in which case \mathbf{x}_* fails to be injective. ■

Example 72 In Example 69 we considered the function $\mathbf{x}(u, v) = (uv, u^2 + v^2, u^3 - v)$. At the point $\mathbf{p} = (2, 1)$ we determined that

$$\mathbf{x}_* \left((x, y)_{\mathbf{p}} \right) = (x + 2y, 4x + 2y, 12x - y)_{(2,5,7)},$$

where (x, y) is the vector part of $\mathbf{v}_{\mathbf{p}}$. This tangent map is clearly injective, but let's compute the rank of the Jacobian of \mathbf{x} at \mathbf{p} to confirm this:

$$\mathcal{J}(\mathbf{x}) = [\nabla(xy) \mid \nabla(x^2 + y^2) \mid \nabla(x^3 - y)] = \begin{bmatrix} y & 2x & 3x^2 \\ x & 2y & -1 \end{bmatrix}$$

hence

$$[2 \ 1] \mathcal{J}(\mathbf{x}) = \begin{bmatrix} 1 & 4 & 12 \\ 2 & 2 & -1 \end{bmatrix}.$$

Note that the first two rows are linearly independent since $\det \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} = -6 \neq 0$. Therefore $\mathcal{J}(\mathbf{x})(2, 1)$ has rank 2.

Let s be some subset of \mathbb{R}^n . It will be necessary to discuss families of tangent vectors whose points of application range over the points in s . Roughly speaking, such a family is called a “tangent vector field” on s . Let \mathcal{U} be an open subset of \mathbb{R}^n . *From now on, all “differentiable” real-valued functions $f : \mathcal{U} \rightarrow \mathbb{R}$ will be assumed to have derivatives of all orders.*

Definition 73 *Let \mathcal{U} be an open subset of \mathbb{R}^n . A vector field V on \mathcal{U} is a function that assigns to each point $\mathbf{p} \in \mathcal{U}$ a tangent vector $V(\mathbf{p}) \in \mathbb{R}_{\mathbf{p}}^n$. If $f : \mathcal{U} \rightarrow \mathbb{R}$ is differentiable, V acts on f in the following way:*

$$V[f](\mathbf{p}) = V(\mathbf{p})[f].$$

The vector field V is said to be differentiable if and only if $V[f] : \mathcal{U} \rightarrow \mathbb{R}$ is differentiable.

Exercise 4.2.1 *Prove Lemma 11.12 in the text.*

Definition 74 *For each $i = 1, \dots, n$, define the i^{th} standard coordinate vector field U_i on \mathbb{R}^n by*

$$U_i(\mathbf{p}) = (\mathbf{e}_i)_{\mathbf{p}}.$$

Taken together, the family of standard coordinate vector fields $\{U_1, \dots, U_n\}$ on \mathbb{R}^n is called the natural frame field of \mathbb{R}^n .

Definition 75 *The natural coordinate functions on \mathbb{R}^n are the functions $u_i : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by*

$$u_i(\mathbf{p}) = p_i$$

where $\mathbf{p} = (p_1, \dots, p_n)$. When $n = 1, 2$, or 3 we often denote the natural coordinate functions by $\{t\}$, $\{u, v\}$, or $\{x, y, z\}$, respectively.

4.3 The Tangent Space

We begin with some notation. Let $\{U_1, \dots, U_n\}$ be the natural frame field on \mathbb{R}^n . Let \mathcal{U} be an open subset of \mathbb{R}^n , let $\mathbf{p} \in \mathcal{U}$, and let $f : \mathcal{U} \rightarrow \mathbb{R}$ be differentiable. Then

$$U_i(\mathbf{p})[f] = (\mathbf{e}_i)_{\mathbf{p}}[f] = \nabla f(\mathbf{p}) \bullet \mathbf{e}_i = \frac{\partial f}{\partial u_i}(\mathbf{p}) = \left(\frac{\partial}{\partial u_i} \Big|_{\mathbf{p}} \right) [f].$$

The symbols $\frac{\partial}{\partial u_i}$ and U_i will be used interchangeably. Thus $\frac{\partial}{\partial u_i} \Big|_{\mathbf{p}} = (\mathbf{e}_i)_{\mathbf{p}}$. In particular, if \mathcal{U} is an open subset of \mathbb{R}^2 , $\mathbf{p} \in \mathcal{U}$, $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ is a patch and $\mathbf{q} = \mathbf{x}(\mathbf{p})$ we have

$$\mathbf{x}_* \left(\frac{\partial}{\partial u} \Big|_{\mathbf{p}} \right) = (\mathbf{x}_u(\mathbf{p}))_{\mathbf{q}} \quad \text{and} \quad \mathbf{x}_* \left(\frac{\partial}{\partial v} \Big|_{\mathbf{p}} \right) = (\mathbf{x}_v(\mathbf{p}))_{\mathbf{q}}. \quad (4.3)$$

If $(x, y)_{\mathbf{p}} \in \mathbb{R}_{\mathbf{p}}^2$, then

$$(x, y)_{\mathbf{p}} = [x\mathbf{e}_1 + y\mathbf{e}_2]_{\mathbf{p}} = x(\mathbf{e}_1)_{\mathbf{p}} + y(\mathbf{e}_2)_{\mathbf{p}} = x \frac{\partial}{\partial u} \Big|_{\mathbf{p}} + y \frac{\partial}{\partial v} \Big|_{\mathbf{p}}.$$

By the linearity of \mathbf{x}_* and 4.3 we have

$$\mathbf{x}_* \left((x, y)_{\mathbf{p}} \right) = x \mathbf{x}_* \left(\frac{\partial}{\partial u} \Big|_{\mathbf{p}} \right) + y \mathbf{x}_* \left(\frac{\partial}{\partial v} \Big|_{\mathbf{p}} \right) = x (\mathbf{x}_u(\mathbf{p}))_{\mathbf{q}} + y (\mathbf{x}_v(\mathbf{p}))_{\mathbf{q}}.$$

So 4.3 tells us that the vector part of tangent vector $\mathbf{x}_*(\mathbf{v}_{\mathbf{p}}) \in \mathbb{R}_{\mathbf{q}}^3$ lies in the 2-plane tangent to the trace of \mathbf{x} at $\mathbf{q} = \mathbf{x}(\mathbf{p})$.

Theorem 76 *Let \mathcal{U} be an open subset of \mathbb{R}^2 , let $\mathbf{p} \in \mathcal{U}$, and let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a patch with $\mathbf{q} = \mathbf{x}(\mathbf{p})$. Then \mathbf{x} is regular at \mathbf{p} if and only if $\mathbf{x}_* : \mathbb{R}_{\mathbf{p}}^2 \rightarrow \mathbb{R}_{\mathbf{q}}^3$ is injective.*

Proof. By Theorem 71 $\mathbf{x}_* : \mathbb{R}_{\mathbf{p}}^2 \rightarrow \mathbb{R}_{\mathbf{q}}^3$ is injective if and only if $\text{rank}(\mathcal{J}(\mathbf{x})(\mathbf{p})) = 2$. By definition, \mathbf{x} is regular if and only if $\text{rank}(\mathcal{J}(\mathbf{x})(\mathbf{p})) = 2$. ■

Theorem 76 tells us that a regular patch \mathbf{x} admits a 2-dimensional vector space of tangent vectors at each of the point in its image.

Definition 77 *Let \mathcal{U} be an open subset of \mathbb{R}^2 , let $\mathbf{p} \in \mathcal{U}$, let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a regular patch, and let $\mathbf{q} = \mathbf{x}(\mathbf{p})$. The tangent space to \mathbf{x} at \mathbf{q} , denoted by $\mathbf{x}(\mathcal{U})_{\mathbf{q}}$, is the image of $\mathbf{x}_* : \mathbb{R}_{\mathbf{p}}^2 \rightarrow \mathbb{R}_{\mathbf{q}}^3$.*

Here is an example of a patch \mathbf{x} that fails to be regular at \mathbf{p} ; we observe that no tangent plane exists at the point $\mathbf{q} = \mathbf{x}(\mathbf{p})$.

Example 78 Define a patch $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by

$$\mathbf{x}(u, v) = (u^2, u^3, v).$$

The trace of \mathbf{x} is the vertical cylinder generated by the semi-cubical parabola $(u^2, u^3, 0)$ in the xy -plane; it has two leaves that meet at a crease along the z -axis. Hence at each point along the z -axis, the trace of \mathbf{x} can have no tangent plane. The following simple calculation bears this out:

$$\mathbf{x}_u = (2u, 3u^2, 0) \text{ and } \mathbf{x}_v = (0, 0, 1).$$

Note that the z -axis is the coordinate curve $\mathbf{x}(0, v) = (0, 0, v)$, where $v \in \mathbb{R}$. So $\mathbf{x}_u(\mathbf{p}) = (0, 0, 0)$ at each point \mathbf{p} whose image lies on the z -axis; it follows that $\mathbf{x}_u(\mathbf{p})$ and $\mathbf{x}_v(\mathbf{p})$ are linearly dependent and fail to span a 2-plane tangent to the trace of \mathbf{x} at $\mathbf{x}(\mathbf{p})$ on the z -axis.

In the discussion above we observed that if $\mathbf{x}_u(\mathbf{p})$ and $\mathbf{x}_v(\mathbf{p})$ are linearly independent, they span the tangent space at the point $\mathbf{q} = \mathbf{x}(\mathbf{p})$. When $n = 3$, the cross product $\mathbf{x}_u(\mathbf{p}) \times \mathbf{x}_v(\mathbf{p})$ is normal to the trace of \mathbf{x} at \mathbf{q} ; by letting \mathbf{p} range over \mathcal{U} , this cross product defines a unit normal vector at each point \mathbf{q} on the trace of \mathbf{x} .

Definition 79 Let \mathcal{U} be an open subset of \mathbb{R}^2 and let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a regular patch. The unit normal vector field (or surface normal) \mathbf{U} on $\mathbf{x}(\mathcal{U})$ is defined by

$$\mathbf{U}(\mathbf{p}) = \left[\frac{\mathbf{x}_u(\mathbf{p}) \times \mathbf{x}_v(\mathbf{p})}{\|\mathbf{x}_u(\mathbf{p}) \times \mathbf{x}_v(\mathbf{p})\|} \right]_{\mathbf{x}(\mathbf{p})},$$

where $\mathbf{p} \in \mathcal{U}$.

Since a patch is regular at \mathbf{p} if and only if $\mathbf{x}_u(\mathbf{p})$ and $\mathbf{x}_v(\mathbf{p})$ are linearly independent, the unit normal vector field \mathbf{U} is defined at each point on the trace $\mathbf{x}(\mathcal{U})$.

Definition 80 Let \mathcal{U} be an open subset of \mathbb{R}^2 and let $\mathbf{x} : \mathcal{U} \rightarrow \mathbb{R}^3$ be a regular patch. Let S^2 denote the unit 2-sphere in \mathbb{R}^3 . The Gauss map of \mathbf{x}

$$U : \mathcal{U} \rightarrow S^2$$

is defined by $U(\mathbf{p}) = \{\text{the vector part of } \mathbf{U}(\mathbf{p})\}$.

Note that the Gauss map U already appeared in our discussion of surface area in chapter 3. We'll use the derivative of the Gauss map to measure the "curvature" of a patch in much the same way that we used the derivative of the unit tangent to measure the curvature of a regular curve

