Periodic Orbits on a Triangular Air Hockey Table

A Senior Thesis Submitted to the Department of Mathematics In Partial Fulfillment of the Requirements for the Departmental Honors Baccalaureate

By Andrew M. Baxter

Millersville, Pennsylvania

May 2005

This Senior Thesis was completed in the Department of Mathematics, defended before and approved by the following members of the Thesis committee

---

Ronald Umble, Ph.D (Thesis Advisor)
Professor of Mathematics
Millersille University

Dorothee Blum, Ph.D
Associate Professor of Mathematics
Millersville University

Dennis DeTurck, Ph.D
Professor of Mathematics,
University of Pennsylvania

Abstract

We explore the existence of periodic orbits on a triangular air hockey table, also known as the billiards problem on a triangle. In section 2, we explore orbits on the equilateral triangle in detail, as well as classify and enumerate them. We summarize results by previous authors concerning more general triangles in section 3.

Periodic Orbits in a Triangular Air Hockey Table

Contents

1 Introduction

Consider a triangular air hockey table. Sending a puck into motion on this table causes it to bounce about the table along a path completely determined by its initial position, angle of departure, and initial speed. If we assume the surface is frictionless and observe the path for an arbitrarily long period of time we can ignore the speed and simplify the situation by starting our observations at the point of the first bounce.

We assume that the action of a puck bouncing off a wall follows the standard rules of reflection: *the angle of incidence equals the angle of reflection.* This rule does not apply when a puck strikes the corner, so we follow convention and decide that a path terminates upon reaching a corner. This problem is also described as a ball bouncing on a billiard table; consequently problems of this sort are commonly known as "billiards problems." Tabachnikov's work "Billiards" [5] provides a thorough compilation of these problems and their known solutions.

Before proceeding, we must define the terminology that will be used for the remainder of the paper.

Definition 1 *An orbit is the path that the puck follows. A terminal orbit is an orbit that terminates at a vertex. A periodic orbit of period n is an orbit that retraces itself after n bounces. An infinite orbit is any orbit that is neither terminal or periodic.*

In this paper we are concerned with the existence and enumeration of periodic orbits on various triangles. In Section 2, we concern ourselves primarily with the enumeration of periodic orbits on the equilateral triangle. We first show the existence of a periodic orbit on the equilateral triangle, a period 3 orbit and the larger collection of period 6 orbits. After showing that the period 3 orbit is the only periodic orbit with odd period, we set about finding orbits of period 2n. We create a coordinate system which facilitates finding vectors which represent periodic orbits. Once we find a formula which allows us to calculate the period of the orbit given a vector, we set about showing that there are $P(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$ vectors which represent period 2n orbits.

A problem arises, however, as this formula counts orbits that may be little more than repetitions of lower-period orbits. The single period 8 orbit counted is actually two repetitions of a period 4 orbit counted as a single unit. We then set out to separate the duplicate-free orbits (those which are not repetitions) from the orbits containing duplicates and enumerate each class. We prove that there exists at least one duplicate-free period $2n$ orbit, granted that $n \neq 1, 4, 6, 10$. Furthermore, if we let $\mu(d)$ be the Möbius function given by

$$\mu(d) = \begin{cases} 1, & d = 1, \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ with } p_i\text{'s distinct primes}, \\ 0, & \text{otherwise} \end{cases}$$

then there are $\sum_{d|n} \mu(d) P\left(\frac{n}{d}\right)$ duplicate-free period $2n$ orbits, where d ranges over all divisors of n.

Section 3 gives an overview of cases in which the following long-standing open problem has been proven true.

Conjecture 2 *Every triangle admits a periodic orbit.*

It must be made clear before proceeding that the periodic orbits discussed are in truth *families* of orbits. Periodic orbits commonly appear in groups sharing an initial angle whose initial points form a connected subset of the initial side. For simplicity, these families will be treated as a single periodic orbit.

2 The Equilateral Triangle

To satisfy Conjecture 2 in the case of the equilateral triangle, one needs only to find a periodic orbit on the triangle. This can be shown by means of a sketch.

Theorem 3 *Any equilateral triangle admits a periodic orbit.*

Proof. The simplest orbit is the period 3 “orthoptic" [2] or "Fagnano" [5] orbit. Starting at any midpoint at a 60° angle, the puck proceeds to bounce on the midpoint of each side, forming the period three orbit shown in Figure 1.

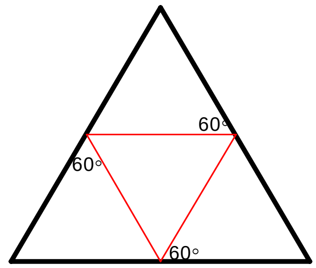


Figure 1: The period 3 orbit.

If the puck does not begin at the midpoint, a period 6 orbit arises, as seen in Figure 2.

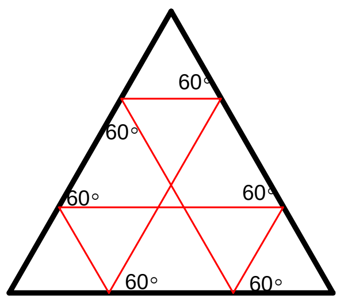


Figure 2: A period 6 orbit.

■

Elementary geometry proves reflecting the triangle about a side is equivalent to bouncing the puck on the side. The "unfolding" action makes orbits much easier to handle.

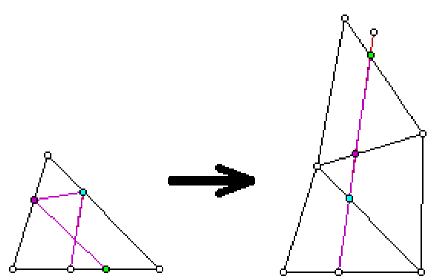


Figure 3: An unfolding.

An orbit is periodic if and only if the puck returns to its initial point at its initial angle. In the unfolded triangles, any line that connects a point to its image is a periodic orbit if the image lies on an edge parallel to the initial edge. To demonstrate, examine the unfolded versions of the period 6 and period 10 orbits in Figure 4.

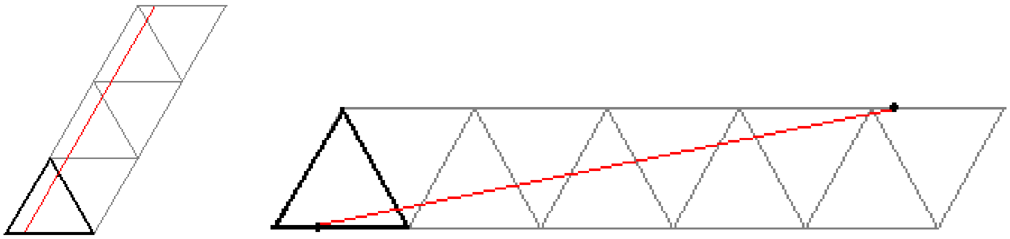


Figure 4: The unfolded period 6 and period 10 orbits.

The converse holds save for a single case. Unfolding the period 3 orbit shows that it does not satisfy the parallel edge requirement. If the trajectory is continued through three more reflections (six in all), one sees that the orbit does eventually reach an image of the original point that lies on a parallel edge.

The symmetry of the equilateral triangle allows us to tessellate the plane by reflecting in its edges and their images. The ability to generate a tessellation in this way is unique to equilateral triangles, since performing multiple reflections in other triangles can lead to overlap. We are interested in the lines in the tessellation and not the triangles themselves. For clarity, an edge is the line segment between two adjacent vertices.

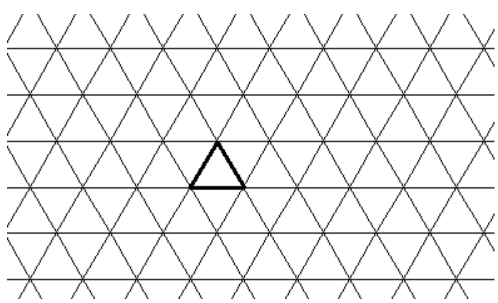


Figure 5: A tessellation of the plane with equilateral triangles.

Any vector drawn on this framework of lines represents an orbit on the equilateral triangle. The task of finding and classifying periodic orbits on the equilateral triangle reduces to finding and classifying vectors that represent periodic orbits. For simplicity, all vectors will have initial points on horizontal edges.

Definition 4 *Let Γ be an orbit with initial point p on edge e and terminal point on line segment m. Then m lies in the bounce circle of radius n centered at p $(n \in \mathbb{N})$ if Γ crosses* *n lines excluding e. The edge e is the bounce circle of radius 0, and the bounce circle of radius ∞ is defined to be the line containing e, minus e itself.*

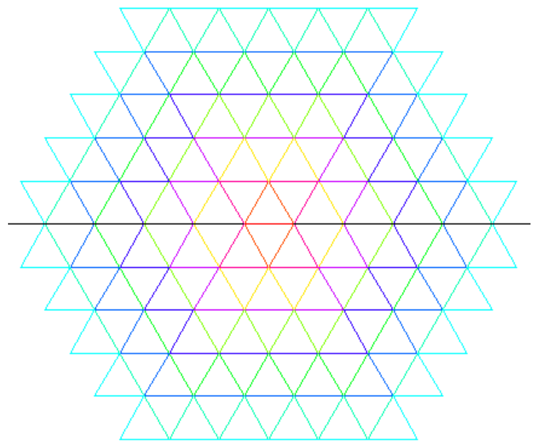


Figure 6: Bounce circles of radii 0 - 10, ∞ (coded by color).

Clearly every edge is in exactly one bounce circle. Note that every horizontal line is contained in a bounce circle with even or infinite radius.

Theorem 5 *The equilateral triangle admits exactly one periodic orbit of odd period, the period 3 orthoptic orbit.*

Proof. Let Γ be the vector representation of a periodic orbit with initial point p and terminal point q on a bounce circle of odd radius. Let θ be the angle formed by Γ and the edge containing p; let α be the angle formed by Γ and the edge containing q. Since Γ is periodic, $\theta = \alpha$. Finding values for θ and α depends on whether q lies on a left- or right-leaning diagonal.

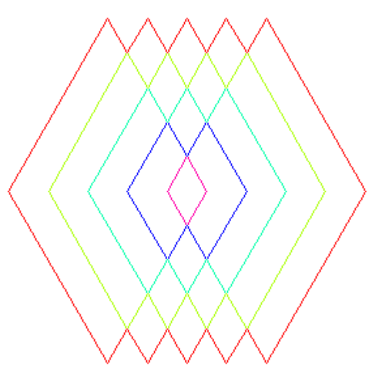


Figure 7: Bounce circles with radii 1,3,5,7 and 9.

Case 1: If q lies on a right-leaning diagonal, then $\theta = \alpha = 30°$. However, $\theta = 30°$ does not yield any crossings with odd bounce circles on right-leaning diagonals, ruling out this case.

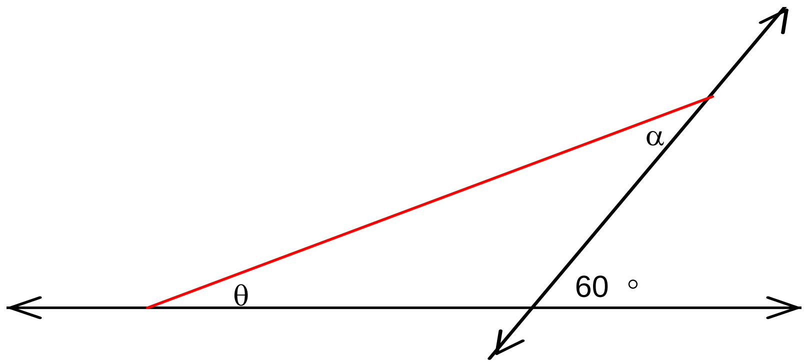


Figure 8: Point Q on a right-leaning diagonal.

Case 2: If q lies on a left-leaning diagonal, then $\theta = \alpha = 60°$. $\theta = 60°$ yields the period 3 orbit from Theorem 3.

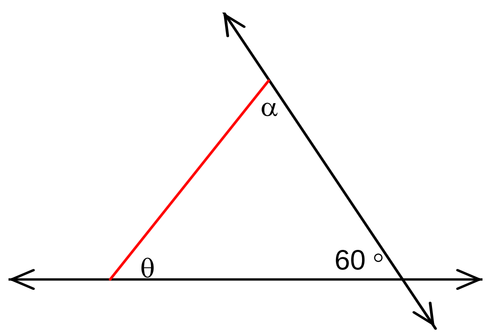


Figure 9: Point Q on a left-leaning diagonal.

■

Since the period 3 orbit is such a special case, we treat it as a degenerate period 6 orbit from this point forward.

There is a natural coordinate system on the equilateral triangle tessellation. Choose initial point p to be the origin, let the x-axis remain horizontal, and let the y-axis be parallel to the right-leaning diagonals. Setting the original triangle as a unit triangle, we obtain the coordinate system shown below.

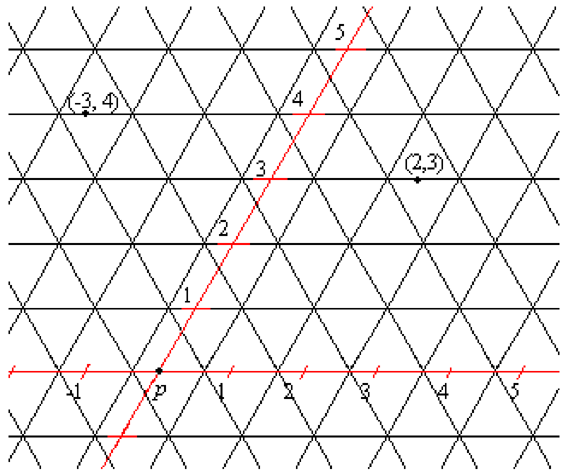


Figure 10: The rhombic coordinate system.

Elementary matrix algebra yields the following change of basis matrices:

From Euclidean to Rhombic:	From Rhombic to Euclidean:
$\begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$

All ordered pairs from this point forward are written as points or vectors in this rhombic basis. Given a vector $\Gamma = (x, y)$, $y > 0$, one can use the following formulas to compute the geometric length L and initial angle θ.

$$L((x, y)) = \sqrt{x^2 + xy + y^2} \tag{1}$$

$$\theta((x, y)) = \arctan\left(\frac{y\sqrt{3}}{2x + y}\right) \tag{2}$$

There is a surprisingly simple method to determine if a given vector represents a periodic orbit.

Theorem 6 *An orbit vector (x, y) is periodic if and only if x and y are integers such that $x \equiv y \pmod 3$.*

Proof. Vector (x, y) is periodic if and only if point (x, y) is the image of the origin after a finite number of reflections and lies on a horizontal edge. We now describe this set of parallel edges.

Highlighting all images of the edge that contains the origin reveals a tessellation of the plane by hexagons.

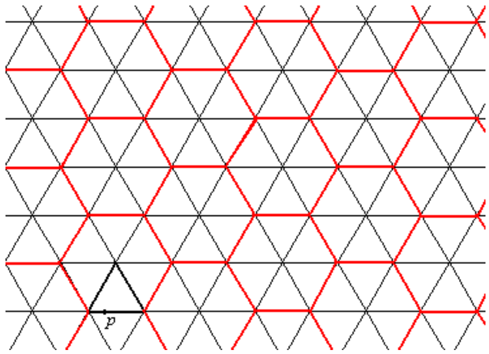


Figure 11: The hexagonal tessellation given by an equilateral tessellation.

We now find two linearly independent vectors which act as a basis for all vectors representing periodic orbits. The vectors $(-1, 2)$ and $(1, 1)$ form such a basis. Both are images of the origin and lie on horizontal edges and so represent periodic orbits. Any image of the origin that lies on a horizontal edge can be written as $a(-1, 2) + b(1, 1)$ for some integers a and b. Since $-1 \equiv 2$ and $1 \equiv 1 (\bmod 3)$, for $a, b \in \mathbb{Z}$ if $(x, y) = a(-1, 2) + b(1, 1)$ (i.e. is a periodic orbit) then $x \equiv y (\bmod 3)$.

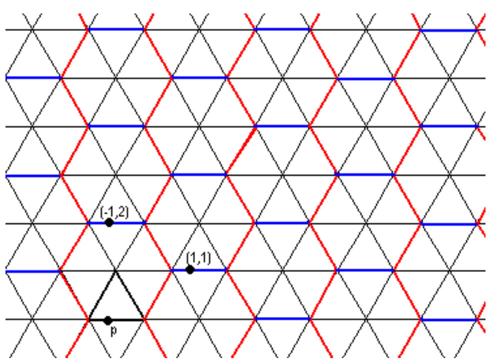


Figure 12: Periodic orbits initiate at P and terminate on a blue edge.

■

Given a vector representing a periodic orbit, we can determine the period by counting the number of lines the vector crosses. The vector of a period n orbit has a terminal point on the bounce circle of radius n, as is expected. The following theorem makes calculating the period very easy.

Theorem 7 *The period of a periodic orbit (x, y), $y > 0$, is given by*

$$Period(x,y) = \begin{cases} 2(x+y), & x \geq 0, y > 0 \\ 2y, & x < 0, y > -x \\ -2x, & x < 0, 0 < y \leq -x \end{cases} \tag{3}$$

Note that when $y < 0$, $Period(x, y) = Period(x, -y)$ by symmetry of the tessellation.

Proof. We derive the formula by considering each of the three cases in turn.

Case 1: Let $x \geq 0, y > 0$.

Let Γ be a periodic orbit from one horizontal to another. Cover Γ with parallelograms bounded by two horizontals and two right-leaning diagonals such that no two parallelograms intersect. A left-leaning diagonal cuts each parallelogram into two equilateral triangles.

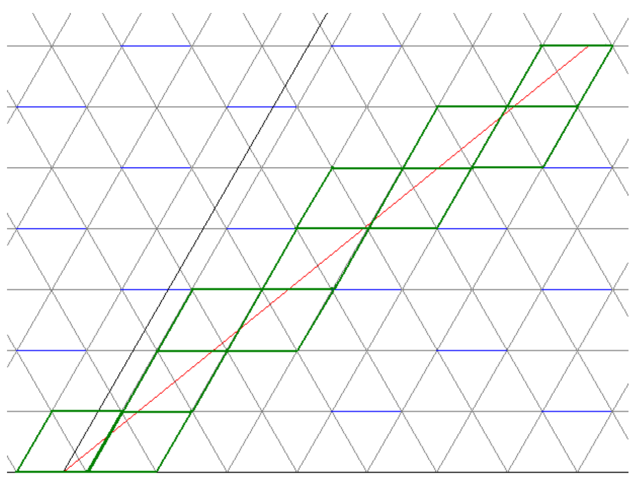


Figure 13: Parallelograms covering the vector $\Gamma = (4, 7)$.

For any parallelogram, Γ must pass through either the left or bottom side (exclusive) and through either the top or right side (exclusive), by the restriction of x and y. Thus, Γ must pass through the left-leaning diagonal contained in the parallelogram. Hence a left-leaning diagonal is crossed for each horizontal and right-leaning diagonal crossed.

Let h, r, and l be the number of horizontals, right-leaning diagonals, and left-leaning diagonals crossed. By inspection, $h = y$ and $r = x$, and we have just shown that $l = h + r$. Therefore,

$$
\begin{aligned}
Period(x,y) &= h + r + l \\
&= y + x + (y + x) \\
&= 2(x + y).
\end{aligned}
$$

<u>*Case 2*:</u> $x < 0, y > -x$.

We cover Γ with parallelograms as in case 1, except the parallelograms are bounded by two right-leaning diagonals and two left-leaning diagonals and split by a horizontal. Furthermore, we must cut one of the parallelograms in half, placing the top half on the original triangle and the bottom half such that its horizontal aligns with the edge containing the terminal point p'. Now one horizontal is crossed for each left- or right-leaning diagonal crossed.

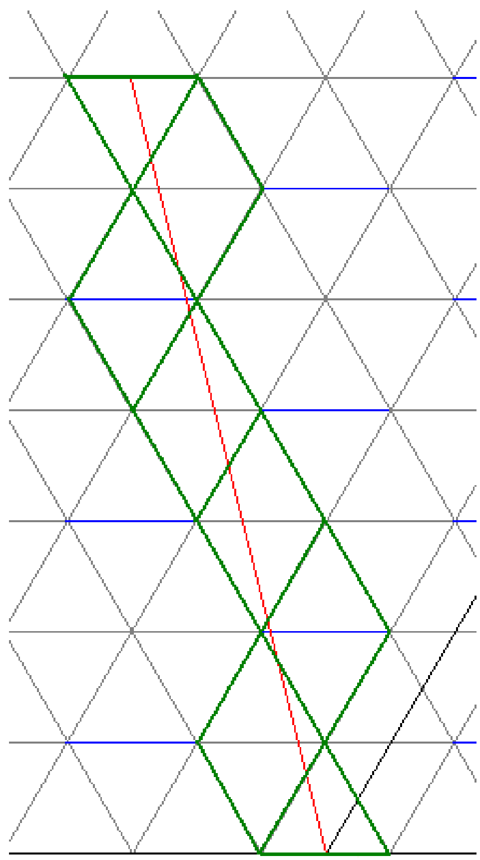


Figure 14: Parallelograms covering the vector $\Gamma = (-5, 7)$.

We now have $h = r + l$, or rather, $l = h - r$. Since $x < 0$, $r = -x$, so

$$
\begin{aligned}
Period(x,y) &= h + r + l \\
&= y - x + (y + x) \\
&= 2y.
\end{aligned}
$$

<u>*Case 3*:</u> $x < 0, 0 < y \leq -x$.

Again, we cover Γ by parallelograms, except the parallelograms are bounded by two left-leaning diagonals and two horizontals and split by a right-leaning diagonals.

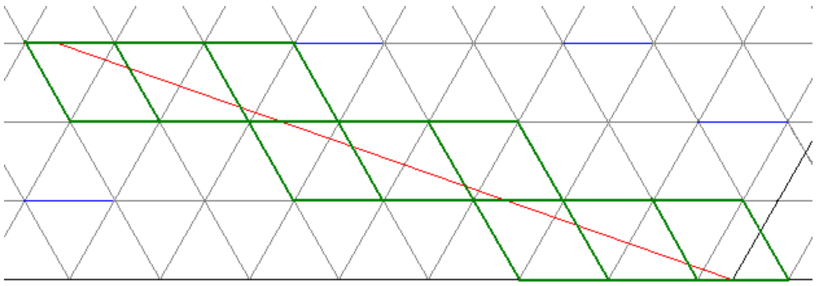


Figure 15: Parallelograms covering the vector $\Gamma = (-9, 3)$.

This makes $r = h + l$, or rather, $l = r - h$. Again, $r = -x$ since $x < 0$. Therefore

$$\begin{aligned} Period(x,y) &= h + r + l \\ &= y - x + (-x - y) \\ &= -2x. \end{aligned}$$

■

Dealing with three period formulas quickly complicates matters. Fortunately we may restrict our focus to a region requiring only one of the three.

Proposition 8 *For any orbit (not necessarily periodic) on the equilateral triangle, there are no more than three different bounce angles, with at least one between* 30° *and* 60°*, inclusive.*

Proof. The tessellation consists of three sets of parallel lines, each intersecting the other two at 60°. Orbit Γ could either run parallel to one of the three sets or cut through all three.

Γ cuts through all three lines when $\alpha < 60^\circ$.

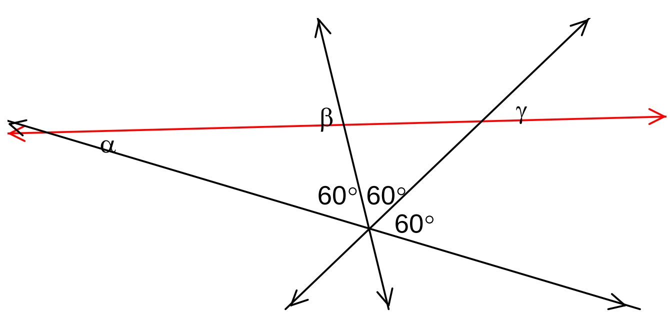


Figure 16: Γ intersects the three lines at angles α, β and γ.

Trigonometry reveals that $\beta = 60^\circ + \alpha$ and $\gamma = 60^\circ - \alpha$, so either α or γ lies in the closed interval $[30^\circ, 60^\circ]$.

When $\alpha = 60^\circ$, Γ runs parallel to one of the diagonals and $\alpha \in [30^\circ, 60^\circ]$. ∎

Remark 9 *We can focus on periodic orbits with initial angle between* 30° *and* 60°*, inclusive. In terms of the coordinates, this outlines the region* $x \geq 0, y \geq x$*.*

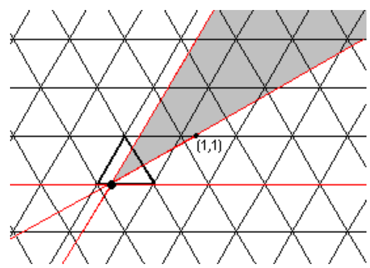


Figure 17: The region $0 \leq x \leq y$.

This raises some interesting questions: Is there a periodic orbit of period $2n$ for any $n \in \mathbb{N}$? If so, how many periodic orbits are there of period $2n$ in all?

In addressing these questions, we make a simplifying assumption: If we repeat a period n orbit k times, we count the entire path as a period kn orbit. This is known as a k-fold duplication of a period n orbit, or a period kn orbit containing k duplicates. For example, if a period 4 orbit is repeated a second time, the whole orbit is counted as a period 8 orbit since it repeats itself after 8 bounces. This is akin to claiming that the function $\tan(x)$, a function of period π, has period 2π. Factoring out these "orbits containing duplicates" is complicated and is addressed at the end of this section.

The question of existence is easy. Obviously there is no period 2 orbit since that requires the triangle to have a pair of parallel sides. Let $n \geq 2$ be a natural number. If n is even, then $\left(\frac{n}{2}, \frac{n}{2}\right)$ is a period $2n$ orbit. If n is odd, then $\left(\frac{n-3}{2}, \frac{n+3}{2}\right)$ forms a period $2n$ orbit.

To illustrate the second question, consider $n = 11$, that is, the period 22 orbits. Two vectors yield period 22 orbits: $(1, 10)$ and $(4, 7)$.

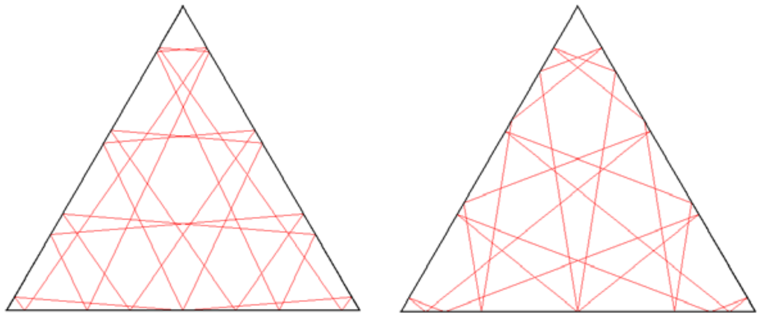


Figure 18: Two period 22 orbits: (1, 10) (left) and (4, 7) (right).

How do we know these two orbits are not permutations of the same orbit? Calculating their lengths by equation 1 shows they are distinct. Alternately, one can calculate bounce angles by equation 2 and Proposition 8 to show they are distinct. We now set about proving the following proposition:

Proposition 10 *Let $\lfloor x \rfloor$ denote the greatest integer function. Given $n \in \mathbb{N}$ such that $n > 1$, there exist $P(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$ period $2n$ orbits.*

This proposition is a special case of a more general combinatorial problem. Given a nonnegative integer n and natural number $m > 1$, how many ways can n be partitioned into two nonnegative numbers $a \leq b$ such that $a + b = n$ and $a \equiv b \pmod{m}$? Proposition 10 considers the case $m = 3$ and $n \geq 2$, as the pairs (a, b) are the vectors for period $2(a+b) = 2n$ orbits.

Definition 11 *Let m and n be integers such that $m \geq 2$ and $n \geq 0$. A partition of n modulo m is a pair of nonnegative integers $a \leq b$ such that $a + b = n$ and $a \equiv b \pmod{m}$. Let $P(n, m)$ denote the number of partitions of n modulo m.*

Theorem 12 *Let m, n, and r be integers such that $m \geq 2$, $n \geq 0$, and r is the least nonnegative residue of $\frac{n(m+1)}{2}$ modulo m. Then*

$$P(n,m) = \begin{cases} 0, & m \text{ even, } n \text{ odd} \\ \left\lfloor \frac{n}{m} \right\rfloor + 1, & m \text{ and } n \text{ even} \\ \left\lfloor \frac{1}{m}\left(\frac{n}{2} - r\right) \right\rfloor + 1, & m \text{ odd.} \end{cases} \tag{4}$$

Proof. Let $m, n \in \mathbb{Z}$ such that $m \geq 2$, $n \geq 0$. Suppose $a + b = n$ and $a \equiv b \pmod{m}$. Then $2a \equiv 2b \equiv n \pmod{m}$. We proceed by cases.

Case 1: m is even, n is odd.

Since $2a \equiv n \pmod{m}$ implies a contradiction, there are no partitions of n modulo m.

Case 2: m and n are even.

Now $2a \equiv 2b \equiv n \pmod{m}$ implies $a \equiv b \equiv \frac{n}{2} \pmod{\frac{m}{2}}$ by modular arithmetic. Any partition of n modulo m has the form $(\frac{n}{2} - \frac{mi}{2}, \frac{n}{2} + \frac{mi}{2})$ for $0 \leq i \leq \frac{n}{m}$. Therefore, there are $\lfloor \frac{n}{m} \rfloor + 1$ partitions of n modulo m.

Case 3: m is odd.

Now $2a \equiv 2b \equiv n \pmod{m}$ implies that $a \equiv b \equiv 2^{-1}n \equiv \frac{m+1}{2}n \pmod{m}$. Let $r \in \{0, 1, 2, \ldots, m-1\}$ such that $r \equiv \frac{m+1}{2}n \pmod{m}$. Then any partition of n modulo m has the form $(r + im, n - (r + im))$ for $0 \leq i \leq \frac{1}{m}\left(\frac{n}{2} - r\right)$. Therefore there are $\lfloor \frac{1}{m}\left(\frac{n}{2} - r\right) \rfloor + 1$ such partitions. ■

See Chart 1 in Appendix B for sample values for $P(n, m)$. This unsightly formula creates surprisingly simple recursive sequences, which in turn have elegant generating functions.

Corollary 13 *Given integer $m \geq 2$, the sequence $P_n = P(n, m)$ has the following recursion relations:*

1. *If m is odd, $P_n = P_{n-2m} + 1$.*
2. *If m is even, $P_{2n} = P_{2n-m} + 1$.*
3. *$P_n = P_{n-2} + \chi_0(n)$, where χ_0 is the characteristic function on the congruence class of 0 modulo m,* $\chi_0(n) = \begin{cases} 1 & n \equiv 0 \mod m \\ 0 & \textit{otherwise.} \end{cases}$

Proof. The first two relations can be derived from the formula for $P(n, m)$ by induction. We will only prove the third, as it is the most general.

Suppose $(a_1, b_1), (a_2, b_2), \ldots, (a_{P_n}, b_{P_n})$ are the partitions of n modulo m. Then $(a_1 + 1, b_1 + 1), (a_2 + 1, b_2 + 1), \ldots, (a_{P_n} + 1, b_{P_n} + 1)$ are each partitions of $n + 2$ modulo m. Additionally, if $n \equiv 0 \pmod{m}$ then $(0, n)$ also partitions n modulo m.

To be sure that all partitions of n modulo m have been counted, suppose (a, b) is a partition of n modulo m, where $a > 0$. Then $(a-1, b-1)$ is a partition of $n-2$ modulo m, so $a-1 = a_i$ and $b-1 = b_i$ for some $i \in \{1, 2, \ldots, P_{n-2}\}$. Thus $P_n = P_{n-2} + \chi_0(n)$. ■

Remark 14 *Every sequence P_n has initial terms $P_0 = 1$ (the partition (0,0)) and $P_1 = 0$ ($0 \neq 1$ modulo $m > 1$).*

Corollary 15 *Given integer $m > 1$, the sequence $P_n = P(n, m)$ has the generating function $\frac{1}{(1-x^2)(1-x^m)}$.*

Proof. Let $m \geq 2$ be a natural number. Define A_n such that

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{(1-x^2)(1-x^m)}. \tag{5}$$

Then $(1-x^2)\sum_{n=0}^{\infty} A_n x^n = \frac{1}{(1-x^m)}$. Distributing on the left and replacing $\frac{1}{(1-x^m)}$ with its generating function $\sum_{n=0}^{\infty} \chi_0(n)x^n$, we are left with

$$\sum_{n=0}^{\infty} \chi_0(n)x^n = \sum_{n=0}^{\infty} A_n x^n - \sum_{n=0}^{\infty} A_n x^{n+2} \tag{6}$$

$$= A_0 + A_1 x + \sum_{n=2}^{\infty} A_n x^n - \sum_{n=2}^{\infty} A_{n-2} x^n \tag{7}$$

$$= A_0 + A_1 x + \sum_{n=2}^{\infty} (A_n - A_{n-2})x^n. \tag{8}$$

Therefore, $A_0 = \chi_0(0) = 1$, $A_1 = \chi_0(1) = 0$, and $A_n - A_{n-2} = \chi_0(n)$ for all $n \geq 2$. By Corollary 13 and Remark 14 with $A_n = P_n$ for all $n \geq 0$, $\frac{1}{(1-x^2)(1-x^m)}$ is the generating function for P_n. ■

The combinatorics-savvy reader will recognize the generating function $\frac{1}{(1-x^2)(1-x^m)}$ as enumerating the number of partitions of n using only 2's and m's as the summands. The bijection between the sets of partitions is discussed in the Appendix.

We are now ready to prove Proposition 10.

Proof. Let $P(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$. We show that $P(n)$ has the same recursion relation and initial conditions as in Corollary 13 and Remark 14 showing $P(n) = P(n, 3)$ for all integers $n \geq 0$.

First confirm by calculation that $P(0) = 1$ and $P(1) = 0$.

Let $n \geq 2$

$$P(n-2) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \tag{9}$$
$$= \left\lfloor \frac{n+2}{2} \right\rfloor - 1 - \left\lfloor \frac{n}{3} \right\rfloor \tag{10}$$
$$= \left\lfloor \frac{n+2}{2} \right\rfloor - \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right). \tag{11}$$

By examination, we see that $\lfloor \frac{n}{3} \rfloor + 1 = \lfloor \frac{n+2}{3} \rfloor + \chi_0(n)$.

$$\left\lfloor \frac{n}{3} \right\rfloor + 1 = \begin{cases} \frac{n}{3} + 1, & n \equiv 0 (\operatorname{mod} 3) \\ \frac{n-1}{3} + 1, & n \equiv 1 (\operatorname{mod} 3) \\ \frac{n-2}{3} + 1, & n \equiv 2 (\operatorname{mod} 3) \end{cases}$$

$$\left\lfloor \frac{n+2}{3} \right\rfloor + \chi_0(n) = \begin{cases} \frac{(n+2)-2}{3} + 1, & n \equiv 0 (\operatorname{mod} 3) \\ \frac{(n+2)}{3} + 0, & n \equiv 1 (\operatorname{mod} 3) \\ \frac{(n+2)-1}{3} + 0, & n \equiv 2 (\operatorname{mod} 3) \end{cases}$$

Therefore, $P(n-2) = P(n) + \chi_0(n)$, so the recursion relation holds. ■

Note that the terminal points of all vectors of period $2n$ orbits lie between the same two left-leaning diagonals.

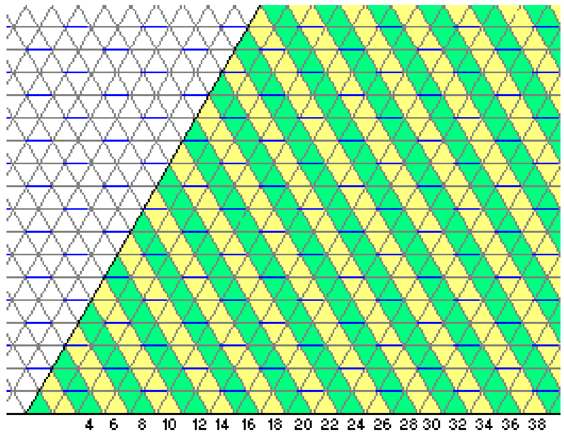


Figure 19: Orbits in a colored band have the same period.

To close this section, we reevaluate the assumption we made when counting orbits of a given period.

Definition 16 *Given periodic orbit* $\Gamma = (x, y)$*, let* $d \in \mathbb{N}$ *be the largest value such that* $x/d \equiv y/d (\text{mod}\, 3)$*. If* $d = 1$*, then* Γ *is a* _duplicate-free orbit_*; otherwise,* Γ *is an* _orbit containing_ d _duplicates_*.*

Finding such a d is cumbersome when x and y are very large. Fortunately the coordinates' common factors determine whether an orbit is duplicate-free.

Theorem 17 *An orbit with vector* (x, y) *is duplicate-free if and only if one of the following is true:*

1. x *and* y *are relatively prime.*

2. *If* $(x, y) = (3a, 3b)$ *for* $a, b \in \mathbb{Z}$*, then* $a \neq b (mod 3)$ *and* a *and* b *are relatively prime.*

Proof. *Part 1:* Suppose (x, y) represents a periodic orbit such that integers x and y are relatively prime. Then the only integer that divides both x and y is 1, and (x, y) must be duplicate-free by Definition 16.

Conversely, suppose (x, y) is duplicate-free and $3 \nmid x$ and $3 \nmid y$. Let $d = \gcd(x, y)$. Then $3 \nmid k$ as well.

$$x \equiv y (\bmod 3) \tag{12}$$

$$dm \equiv dn (\bmod 3), \text{ for some } m, n \in \mathbb{Z} \tag{13}$$

$$m \equiv n (\bmod 3) \tag{14}$$

Then $(m, n) = (\frac{x}{d}, \frac{y}{d})$ is periodic, but since (x, y) is duplicate-free, $d = 1$. Therefore x and y are relatively prime.

Part 2: Suppose $(x, y) = (3a, 3b)$ represents a periodic orbit for integers a and b such that $a \neq b$ (mod 3) and a and b are relatively prime. Then $(x, y) = d\left(\frac{x}{d}, \frac{y}{d}\right)$ for $d \in \mathbb{N}$ implies $d = 1$ or $d = 3$. If $d = 3$, then $\left(\frac{x}{d}, \frac{y}{d}\right) = (a, b)$, which is not a periodic orbit since $a \neq b$ (mod 3). $d = 1$ implies that (x, y) is duplicate-free. Therefore, (x, y) is duplicate-free.

Now suppose $(x, y) = (3a, 3b)$ is duplicate-free. Then (a, b) is not periodic, so $a \neq b$ (mod 3). Secondly, suppose $d \mid a$ and $d \mid b$. Since $\frac{3a}{d} \equiv \frac{3b}{d}$ (mod 3), the orbit $(\frac{x}{d}, \frac{y}{d})$ is periodic. But (x, y) is duplicate-free, however, so $d = 1$. ■

We first answer whether a duplicate-free period $2n$ orbit exists for all n, then conclude this section by giving an exact count.

Theorem 18 *Let $n \geq 2$ be a natural number. Then there exists a duplicate-free period $2n$ orbit if and only if $n \neq 4, 6,$ or 10.*

Proof. Recall $\gcd(x, y) \mid (y - x)$.

Case 1: n is odd.

The duplicate-free orbit is $\left(\frac{n-3}{2}, \frac{n+3}{2}\right)$. A quick check confirms that $\frac{n-3}{2} \equiv \frac{n+3}{2}$ (mod 3). Now $\frac{n+3}{2} - \frac{n-3}{2} = 3$, so if $\frac{n-3}{2}$ is not a multiple of 3 then $\gcd(\frac{n-3}{2}, \frac{n+3}{2}) = 1$.

If $n = 3m$ for some (odd) integer m then $(\frac{n-3}{2}, \frac{n+3}{2}) = \left(3\left(\frac{m-1}{2}\right), 3\left(\frac{m+1}{2}\right)\right)$. Since $a = \frac{m-1}{2}$ and $b = \frac{m+1}{2}$ are consecutive integers, $\gcd(a, b) = 1$ and $a \neq b$ (mod 3). Therefore $\left(\frac{n-3}{2}, \frac{n+3}{2}\right)$ is duplicate-free.

Case 2: $n \equiv 0$ (mod 4)

If $n \neq 4$, the duplicate-free orbit is $\left(\frac{n}{2} - 3, \frac{n}{2} + 3\right)$. This is obviously a periodic orbit. $\frac{n}{2} + 3 - (\frac{n}{2} - 3) = 6$, so if n is not a multiple of 3 then the two coordinates are relatively prime. Note that even though 2 divides the difference, 2 cannot divide either coordinate since both are odd.

If n is a multiple of 3, then $n = 12m$ for some $m \in \mathbb{Z}$, $\frac{n}{2} - 3 = 3(2m-1)$, and $\frac{n}{2} + 3 = 3(2m+1)$. Since $a = 2m-1$ and $b = 2m+1$ are consecutive odd integers, $\gcd(a,b) = 1$ and $a \neq b \pmod 3$.

If $n = 4$, then $x = \frac{n}{2} - 3 = -1$, which is outside the bounds we set in Remark 9. Upon inspection, we find (2,2) is the only mod-3 partition of 4, which is twice the orbit (1,1). Thus there is no periodic orbit of period $2 \times 4 = 8$.

<u>*Case 3*:</u> $n \equiv 2 \pmod 4$

If $n \geq 14$, the duplicate-free orbit is $\left(\frac{n}{2} - 6, \frac{n}{2} + 6\right)$. $\frac{n}{2} + 6 - (\frac{n}{2} - 6) = 12$. Since both coordinates are odd, if $\frac{n}{2} - 6$ is not an multiple of 3, then $\gcd(\frac{n}{2} - 6, \frac{n}{2} + 6) = 1$.

If $\frac{n}{2} - 6$ is a multiple of three, then $n = 6m$ for some (odd) $m \in \mathbb{Z}$. Then $\frac{n}{2} - 6 = 3(m-2)$ and $\frac{n}{2} + 6 = 3(m+2)$. Since $(m+2) - (m-2) = 4$ and m is odd, $\gcd(m-2, m+2) = 1$ and $m - 2 \neq m + 2 \pmod 3$.

If $n = 2, 6$, or 10, $x = \frac{n}{2} - 6$, lies outside the bound $x \geq 0$. By listing partitions modulo 3 for $n = 6$ and $n = 10$ as in case 2, we see that neither has duplicate-free periodic orbits. The case $n = 2$ is special, however, as it does have the duplicate-free periodic orbit (1,1). ■

Enumerating duplicate-free period $2n$ orbits is most easily done by counting periodic orbits containing duplicates and subtracting this number from $P(n)$. We begin with a lemma.

Lemma 19 *A period $2n$ orbit contains a d-fold duplicate only if $d \mid n$. Each duplicate has period $\frac{2n}{d}$.*

Proof. If period $2n$ orbit (x, y) contains d duplicates, then $\left(\frac{x}{d}, \frac{y}{d}\right)$ is a periodic orbit (though not necessarily duplicate-free). Since $d \mid x$ and $d \mid y$, $d \mid (x+y) = n$. Furthermore,

$$\mathit{Period}\left(\frac{x}{d}, \frac{y}{d}\right) = 2\left(\frac{x}{d} + \frac{y}{d}\right) = 2\frac{(x+y)}{d} = \frac{2n}{d}. \tag{15}$$

■

From Proposition 10, there are $P\left(\frac{n}{d}\right)$ period $2n$ orbits containing d duplicates. By examining n's prime factors we can account for all orbits containing duplicates.

Let $n = p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}$, where $p_1, p_2, \dots, p_m$ are prime. Then there are $P\left(\frac{n}{p_1}\right)$ p_1-fold duplicates, $P\left(\frac{n}{p_2}\right)$ p_2-fold duplicates, down to $P\left(\frac{n}{p_m}\right)$ p_m-fold duplicates. However $(p_1 p_2)$-fold duplicates, for example, have been counted twice, so we must subtract $\sum P\left(\frac{n}{p_i p_j}\right)$ for all pairs $i < j$. Now we have subtracted *too many* and must add $(p_i p_j p_k)$-fold duplicates $i < j < k$ back in. Continue this process of alternately adding and subtracting as per the Principle of Inclusion-Exclusion until reaching $p_1 p_2 \cdots p_m$-fold duplicates. The final tally of the $2^m - 1$ terms is the total number of period $2n$ orbits containing duplicates. Let

$$D(n) = P(n) - \sum_{d|n} \mu(d) P\left(\frac{n}{d}\right),$$

where μ is the Möbius function given by

$$\mu(d) = \begin{cases} 1, & d = 1 \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ with } p_i\text{'s distinct primes} \\ 0, & \text{otherwise.} \end{cases}$$

We have proved:

Theorem 20 *There are $D(n)$ period $2n$ orbits containing duplicates.*

Most importantly, we can now count the number of period $2n$ duplicate-free orbits.

Corollary 21 *There are $\sum_{d|n} \mu(d) P\left(\frac{n}{d}\right)$ duplicate-free period $2n$ orbits.*

Proof. Since every orbit is either duplicate-free or duplicate, there are

$$F(n) = P(n) - D(n) \tag{16}$$

duplicate-free period $2n$ orbits. Thus,

$$F(n) = P(n) - \left(P(n) - \sum_{d|n} -\mu(d) P\left(\frac{n}{d}\right)\right) \tag{17}$$

$$= \sum_{d|n} \mu(d) P\left(\frac{n}{d}\right) \tag{18}$$

■

Example 22 *Let* $n = 50 = 2 \cdot 5^2$*. There are*

$$D(n) = P\left(\frac{50}{2}\right) + P\left(\frac{50}{5}\right) - P\left(\frac{50}{2 \cdot 5}\right) \tag{19}$$

$$= P(25) + P(10) - P(5) = 4 + 2 - 1 = 5 \tag{20}$$

orbits containing duplicates. Listing all $P(50) = 9$ *orbits, mark those that contain 2-, 5-, and 10-fold duplicates.*

(x, y)	*2-fold*	*5-fold*	*10-fold*	*Duplicate-free*
$(1, 49)$				✓
$(4, 46)$	✓			
$(7, 43)$				✓
$(10, 40)$	✓	✓	✓	
$(13, 37)$				✓
$(16, 34)$	✓			
$(19, 31)$				✓
$(22, 28)$	✓			
$(25, 25)$		✓		

The table's columns illustrate there are 4 orbits that contain 2-fold duplicates, 2 orbits with 5-fold duplicates, and 1 with both 2- and 5-fold duplicates resulting in a 10-fold duplicate.

Example 23 *Let* $n = 44100 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$*. Then there are*

$$P\left(\frac{44100}{2}\right) + P\left(\frac{44100}{3}\right) + P\left(\frac{44100}{5}\right) + P\left(\frac{44100}{7}\right)$$
$$- \left(P\left(\frac{44100}{2 \cdot 3}\right) + P\left(\frac{44100}{2 \cdot 5}\right) + P\left(\frac{44100}{2 \cdot 7}\right) + P\left(\frac{44100}{3 \cdot 5}\right) + P\left(\frac{44100}{3 \cdot 7}\right) + P\left(\frac{44100}{5 \cdot 7}\right)\right)$$
$$+ \left(P\left(\frac{44100}{2 \cdot 3 \cdot 5}\right) + P\left(\frac{44100}{2 \cdot 3 \cdot 7}\right) + P\left(\frac{44100}{2 \cdot 5 \cdot 7}\right) + P\left(\frac{44100}{3 \cdot 5 \cdot 7}\right)\right)$$
$$- P\left(\frac{44100}{2 \cdot 3 \cdot 5 \cdot 7}\right)$$
$$= 3676 + 2451 + 1471 + 1051 - 1226 - 736 - 526 - 491 - 351 - 211$$
$$+ 246 + 176 + 106 + 71 - 36$$
$$= 5671$$

period 88200 orbits containing duplicates, and

$$P(44100) - 5671 = 7351 - 5671 = 1680$$

duplicate-free orbits.

This formula has a few interesting special cases. For example,

Corollary 24 *There are no period $2p$ orbits containing duplicates if and only if p is prime.*

Proof. This follows directly from $P\left(\frac{p}{p}\right) = P(1) = 0$ and the fact that 1 is the only value of n such that $P(n) = 0$. ■

Chart 2 and Graph 1 in Appendix B show sample values for $P(n), D(n)$, and $F(n)$.

3 Generalization

We now summarize previous work on the existence of periodic orbits on general triangles. As sometimes occurs in research, some results were discovered by the author independently before discovering previously existing literature. Theorems 25, 26, and 27 were discovered independently, as well as a less general form of Theorem 31.

Since the equilateral triangle is both acute and isosceles, one expects to find analogues of the results for equilateral triangle periodic orbits on each of these triangles. The period 6 orbit on the acute triangle corresponds to the orbit (0,3) on the equilateral triangle and even includes the special period 3 case. While Fagnano [2] discovered the period 3 orbit in 1775, I will present my own proof for the period 6.

Theorem 25 *Every acute triangle admits a period 6 orbit.*

Proof. Let $\triangle ABC$ be acute and let $\alpha = \angle CAB, \beta - ABC$, and $\gamma = \angle BCA$. Reflect in the following sequence of lines, where X' represents the reflected image of X: $\overline{BC}, \overline{A'C}, \overline{A'B'}$, $\overline{B'C'}, \overline{A''C'}$ (see Figure 20). Extend segments $\overline{AB}$ and $\overline{A'B'}$ such that they meet at point D, and likewise extend $\overline{A''B''}$ and $\overline{A'B'}$ to meet at E. Now $m\measuredangle ADE = m\measuredangle B''ED = \alpha + \beta - \gamma$, so by alternate interior angles, we see that $\overline{AB}$ and $\overline{A''B''}$ are parallel. Any line contained in all six triangles connecting a point on $\overline{AB}$ to its image on $\overline{A''B''}$ is a periodic orbit. We can draw at least one such line granted that $\gamma < 90°$, which is true by assumption. The "folded" orbit appears as in Figure 21. To generate the period 3 orbit, called the "orthoptic orbit" by Fagnano, connect the intersections of the altitudes and edges (see Figure 22). ■

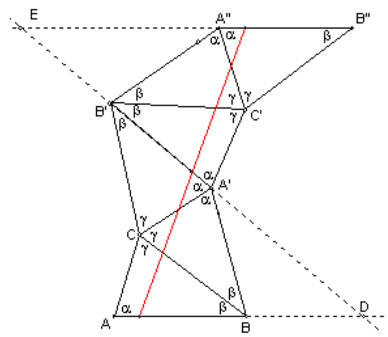


Figure 20: Unfolded period 6 orbit.

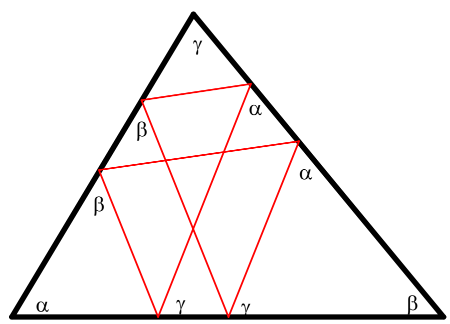


Figure 21: Folded period 6 orbit.

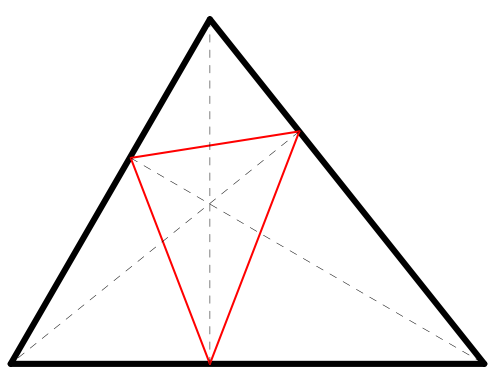


Figure 22: Fagnano's orthoptic period 3 orbit.

The equilateral triangle's period 4 orbit (1,1) has an analogue in any isosceles triangle.

Theorem 26 *Every isosceles triangle admits a period 4 orbit.*

Proof. Let $\triangle ABC$ be isosceles with congruent base angles $\angle CAB$ and $\angle CBA$. Let $\triangle A'BC$ be the reflection of $\triangle ABC$ in $\overline{BC}$; let $\triangle A'BC'$ be the reflection of $\triangle A'BC$ in $\overline{A'B}$; and let $\triangle A'B'C'$ be the reflection of $\triangle A'BC'$ in $\overline{A'C'}$ (see Figure 23)

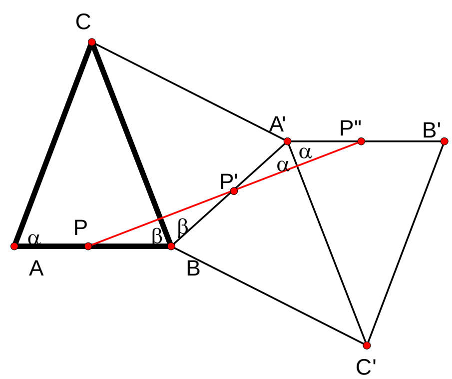


Figure 23: Unfolded period 4 orbit.

Since $\overline{A'B}$ is a transversal and $\angle ABA' \cong \angle BA'B'$, $\overline{AB} \parallel \overline{A'B'}$. Then connecting any point $P \in \overline{AB}$ to its image under the three above reflections P'' yields the period 4 orbit shown in Figure 24.

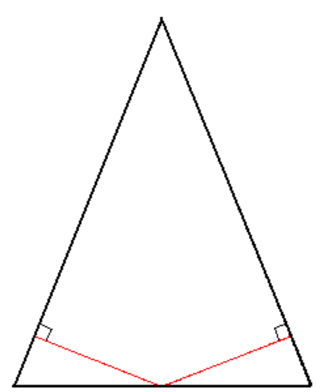


Figure 24: Folded period 4 orbit.

We can state a partial converse since $\overline{AB} \parallel \overline{A'B'}$ if and only if $\angle ABC \cong \angle BAC$. This is not enough to say that no triangle other than an isosceles admits a period 4 triangle, merely that *this sequence* of reflections produces a periodic orbit only if $\triangle ABC$ is isosceles. ■

Since the proof of Theorem 26 depends only on the congruence of the base angles of an isosceles triangle, the conclusion holds for acute, right, and obtuse triangles.

While the orbit in Figure 24 may appear to have only two lines, careful tracing will reveal that it is in fact period 4 as the puck retraces its path in the opposite direction. Orbits that retrace themselves in this manner are easier to discover as their existence is determined by the appearance of two right angles. With the except of Theorem 30 these perpendicular orbits, as they are called, appear in each of the remaining proofs of this section.

Theorem 27 *Every right triangle admits a period 6 orbit.*

Proof. Let $\triangle ABC$ be a right triangle with right angle $\angle ACB$. Let $\triangle AB'C$ be the reflection of $\triangle ABC$ in $\overline{AC}$ and let $\triangle A'B'C$ be the reflection of $\triangle AB'C$ in $\overline{B'C}$. See Figure 25.

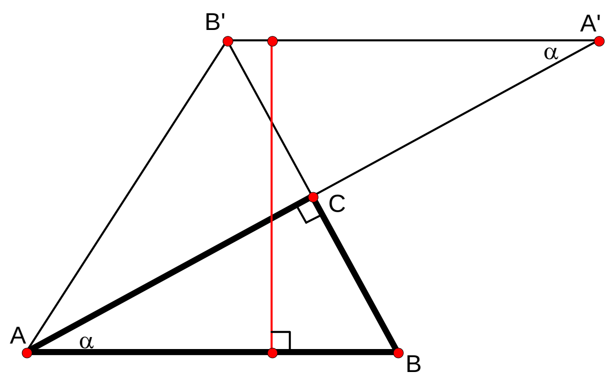


Figure 25: Half of an unfolded period 6 orbit.

Since $\angle ACB$ is a right angle, $\angle ACA'$ is a straight angle. Therefore $\overline{BB'}$ is a transversal between $\overline{AB}$ and $\overline{A'B'}$ forming alternate interior angles $\angle ABC \cong \angle A'B'C$. Hence $\overline{AB} \parallel \overline{A'B'}$, so any common perpendicular generates a period 6 orbit like the one in Figure 26. ■

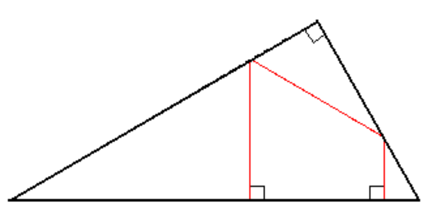


Figure 26: A period 6 orbit.

Remark 28 *This same orbit is obtained if you add a wall along the altitude of an isosceles triangle and begin the period 4 orbit at some point other than the midpoint.*

We are now left with only scalene obtuse triangles. Periodic orbits do not appear so readily here, however. We must carve the set of scalene obtuse triangles into families of triangles sharing some characteristic we can exploit to find a periodic orbit. The first major result is due from Masur [3]; we omit the proof.

Definition 29 *A* <u>*rational polygon*</u> *is a polygon whose interior angles are each a rational multiple of* π.

Theorem 30 *(Masur) Every rational polygon admits a periodic orbit. In fact, there is a dense set of initial angles, each of which yields a periodic orbit.*

Although Masur gives no indication of how to construct them, it follows that every rational triangle admits a periodic orbit. Theorems 25, 26, and 27 hold for both rational and irrational triangles, but we are left with irrational scalene obtuse triangles to explore. Three types of triangles, each possessing a special relationship between the acute angles, have been proven to admit periodic orbits.

Theorem 31 *(Vorobets, et al.) Let* α *and* β *be the acute angles of an obtuse triangle such that* $m\alpha = n\beta < \pi/2$ *for some natural numbers* m *and* n*. Then the triangle admits a period* $2(m+n)$ *orbit.*

Proof. Reflect the triangle $m-1$ times counter-clockwise about α and n times clockwise about β as per Figure 27. This produces a pair of alternate interior angles of measure $m\alpha = n\beta$, and thus a pair of parallel lines. A common perpendicular drawn between these lines represents a periodic orbit. Counting the number of lines crossed and doubling for the return trip, we get $2(m+n)$. The initial angle will be $\pi/2 - m\alpha$. ∎

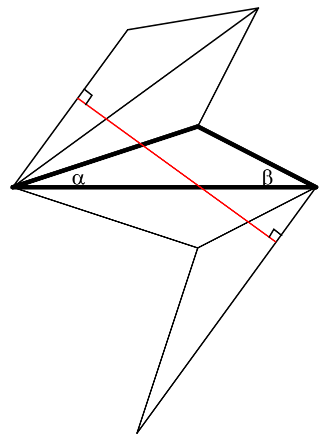


Figure 27: The reflections and common perpendicular when $m = 3$ and $n = 2$.

Remark 32 *This is a generalization of Theorem* 26*, since we obtain an the period 4 orbit on an isosceles triangle when* $m = n = 1$*.*

Theorem 33 *(Vorobets, et al.) Let* α *and* β *be the acute angles of an obtuse triangle such that* $k\alpha+\beta = \pi/2$ *(*$k \in \mathbb{N}$*) and* $\alpha+2\beta > \pi/2$*. Then there exists a period* $4k+2$ *perpendicular orbit.*

Proof. Let $\triangle ABC$ be obtuse with acute angles $\alpha = m\angle CAB$ and $\beta = m\angle ABC$. Reflect $\triangle ABC$ in $\overline{AC}$ to form $\triangle AB'C$; reflect $\triangle AB'C$ in $\overline{AB'}$ to form $\triangle AB'C'$ and so on for a total of $k-1$ reflections as in Figure 28.

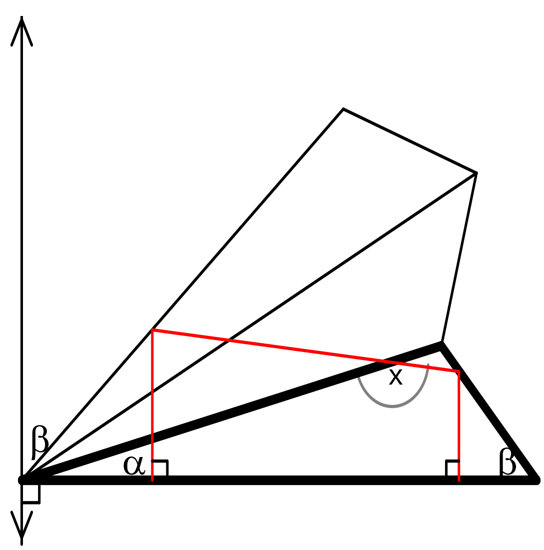


Figure 28: Reflections and half of the orbit when $k = 3$.

Starting at an right angle from side $\overline{AB}$ produces the periodic orbit shown. $k\alpha+\beta = \pi/2$ ensures that the puck returns to side $\overline{AB}$ at a right angle. The condition that $\alpha + 2\beta > \pi/2$ exists so that the angle marked x is less than 180°.

To calculate the period for a general k, consider the three segments making up the orbit. The left vertical crosses k lines excluding $\overline{AB}$. The diagonal segment also crosses the same k lines as well as $\overline{BC}$. One of these ($\overline{AC'}$ in the diagram) has been counted already, so we count k new lines. Looking at the right vertical, only 2 lines are crossed: $\overline{BC}$ and $\overline{AB}$. However, we have counted $\overline{BC}$ with the diagonal leaving us to count only $\overline{AB}$. Therefore in one half-period we cross $2k + 1$ lines, making $2(2k + 1) = 4k + 2$ bounces in the entire circuit. ■

Remark 34 *These triangles are known as "quasi-right triangles" since the triangle is right when $k = 1$. Theorem 27 is the special case of this theorem when $k = 1$.*

Theorem 35 *(Halbeisen and Hungerbuhler) Let α and β be the acute angles of an obtuse triangle such that $m\alpha + n\beta = \pi$ for natural numbers m and n. Then there exists a periodic orbit of period $2(m + n - 1)$, provided*

$$\frac{\pi}{2m}\left(1 - \frac{1}{n-1}\right) < \alpha < \frac{\pi}{2m}\left(1 + \frac{1}{m-1}\right) \tag{21}$$

Proof. Let α and β be angles of a triangle such that $m\alpha + n\beta = \pi$. Reflecting $k - 1$ times around angle α and $n - 1$ times around angle β as in Figure 29, we construct a set of parallel lines, provided one of two conditions holds:

1. $m\alpha \leq \frac{\pi}{2}$ and $\frac{\pi}{2} - m\alpha < \beta$, or

2. $m\alpha > \frac{\pi}{2}$ and $\frac{\pi}{2} - n\beta < \alpha$.

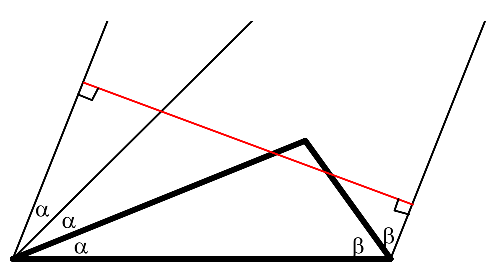


Figure 29: Reflections and common perpendicular when $m = 3$, $n = 2$ and Condition 1 holds.

From the equivalence $m\alpha + n\beta = \pi$, we find that $\pi/2 - m\alpha < \beta$ is equivalent to $(n-2)\pi < 2m\alpha(n-1)$. We also deduce that $\pi/2 - n\beta < \alpha$ is equivalent to $(m-2)\pi < 2n\beta(m-1)$. These two can be solved for α by substituting $\beta = \frac{\pi - m\alpha}{n}$ and combined to form the inequality in equation 21. ■

4 Conclusion

We have shown in Theorem 18 that the equilateral triangle has at least one duplicate-free period $2n$ orbit for every even integer n except 1, 4, 6, and 10, and more precisely in Corollary 21 that there are $\sum_{d|n} \mu(d) P\left(\frac{n}{d}\right)$ duplicate-free period $2n$ orbits. Any function to describe the number of period $2n$ orbits containing duplicates (which could be used to find the number of duplicate-free orbits) that does not rely on divisors of n, one resembling $P(n)$, would need to have roots at all prime numbers and only at prime numbers. The existence of such a function is not known.

While many types of triangles have been shown to admit periodic orbits, gaps in the proof of Conjecture 2 remain. For example, consider the triangle with interior angles $\frac{1}{\pi}$, $\frac{1}{\pi^2}$, and $\frac{\pi^2-\pi-1}{\pi^2}$ radians. This is a scalene obtuse triangle that does not fit the hypotheses of Theorems $30, 31, 33$ or 35. To the author's knowledge, no periodic orbit is known to exist on this triangle, nor has it been shown that one does not exist. A proof that no periodic orbit exists on this or some other triangle would provide a counterexample to Conjecture 2.

Just as the period 6 orbit on the equilateral triangle has an analogue on any acute triangle and the period 4 orbit has an analogue on any isosceles triangle, some other orbits on the equilateral triangle can be transferred to acute isosceles triangles. For example, the isosceles acute triangle has been shown to admit a period 10 orbit with the same unfolding as the orbit (4,1) on the equilateral triangle. It is possible that many of the results in Section 2 could find generalizations that apply to all acute isosceles triangles.

5 Appendices

5.1 Appendix A: A Bijection

Recall Definition 11 and the remarks following Corollary 15. We describe a bijection between the partitions modulo m of nonnegative integer n and the partition of n into summands 2 and m, here denoted $(2, m)$-partitions.

Let (a, b) be a partition of n modulo m. Without loss of generality, let $a \leq b$. Then $a \times 2 + \frac{(b-a)}{m} \times m$ is a $(2, m)$-partition, since $m \mid (b - a)$.

For the inverse, let $a \times 2 + b \times m$ be a $(2, m)$-partition of n. Then $(a, a+bm)$ is a partition of n modulo m.

The bijection is easiest seen with a Ferrers diagram. Consider the case $n = 8$ and $m = 3$. The partitions modulo m are (1,7) and (4,4), whereas the $(2, m)$-partitions are 2+3+3 and 2+2+2+2.

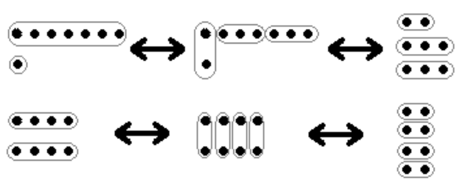


Figure 30: Ferrers diagram for $n = 8$ and $m = 3$.

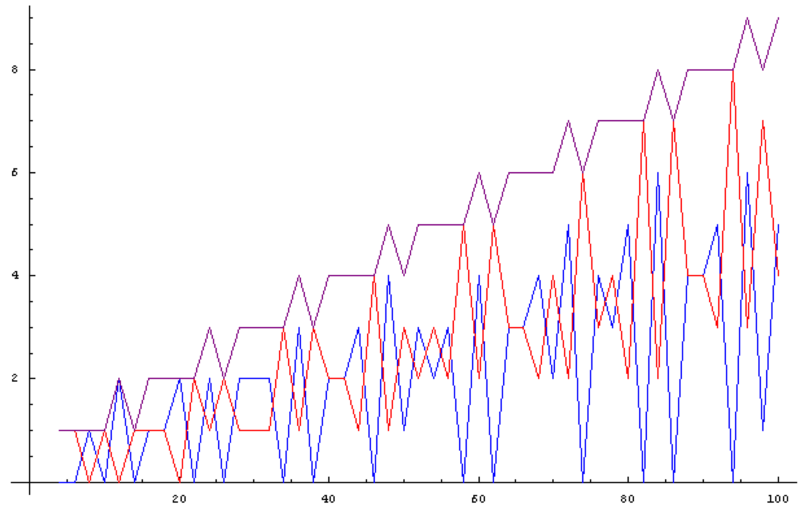
5.2 Appendix B: Charts and Graphs

n \ m	2	3	4	5	6	7	8	n \ m	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	20	11	4	6	3	4	2	3
1	0	0	0	0	0	0	0	21	0	4	0	2	0	2	0
2	2	1	1	1	1	1	1	22	12	4	6	3	4	2	3
3	0	1	0	0	0	0	0	23	0	4	0	2	0	2	0
4	3	1	2	1	1	1	1	24	13	5	7	3	5	2	4
5	0	1	0	1	0	0	0	25	0	4	0	3	0	2	0
6	4	2	2	1	2	1	1	26	14	5	7	3	5	2	4
7	0	1	0	1	0	1	0	27	0	5	0	3	0	2	0
8	5	2	3	1	2	1	2	28	15	5	8	3	5	3	4
9	0	2	0	1	0	1	0	29	0	5	0	3	0	2	0
10	6	2	3	2	2	1	2	30	16	6	8	4	6	3	4
11	0	2	0	1	0	1	0	31	0	5	0	3	0	2	0
12	7	3	4	2	3	1	2	32	17	6	9	4	6	3	5
13	0	2	0	1	0	1	0	33	0	6	0	3	0	2	0
14	8	3	4	2	3	2	2	34	18	6	9	4	6	3	5
15	0	3	0	2	0	1	0	35	0	6	0	4	0	3	0
16	9	3	5	2	3	2	3	36	19	7	10	4	7	3	5
17	0	3	0	2	0	1	0	37	0	6	0	4	0	3	0
18	10	4	5	2	4	2	3	38	20	7	10	4	7	3	5
19	0	3	0	2	0	1	0	39	0	7	0	4	0	3	0

Chart 1: Sample values for $P(n, m)$

n	2n	P(n)	F(n)	D(n)	n	2n	P(n)	F(n)	D(n)
2	4	1	1	0	21	42	4	2	2
3	6	1	1	0	22	44	4	1	3
4	8	1	0	1	23	46	4	4	0
5	10	1	1	0	24	48	5	1	4
6	12	2	0	2	25	50	4	3	1
7	14	1	1	0	26	52	5	2	3
8	16	2	1	1	27	54	5	3	2
9	18	2	1	1	28	56	5	2	3
10	20	2	0	2	29	58	5	5	0
11	22	2	2	0	30	60	6	2	4
12	24	3	1	2	31	62	5	5	0
13	26	2	2	0	32	64	6	3	3
14	28	3	1	2	33	66	6	3	3
15	30	3	1	2	34	68	6	2	4
16	32	3	1	2	35	70	6	4	2
17	34	3	3	0	36	72	7	2	5
18	36	4	1	3	37	74	6	6	0
19	38	3	3	0	38	76	7	3	4
20	40	4	2	2	39	78	7	4	3

Chart 2: Sample values for the total number of orbits ($P(n)$), duplicate-free orbits ($F(n)$), and orbits containing duplicates ($D(n)$).



Graph 1: Plot of Chart 2. $P(n)$ is shown in purple, $F(n)$ in red, and $D(n)$ in blue.

References

[1] Gutkin, E. *Billiards in Polygons.* Physica 19D. pp 311-333. 1986.

[2] Halbeisen, L. and N. Hungerbuhler. *On Periodic Billiard Trajectories in Obtuse Triangles.* SIAM Review. Vol. 42, No. 4, pp 657-670. 2000.

[3] Masur, H. *Closed Trajectories For Quadratic Differentials With an Application to Billiands.* Duke Mathematical Journal. Vol. 53, No. 2, pp 307-313. 1986.

[4] Rosen, K.H. *Elementary Number Theory and Its Applications.* Addison Wesley Longman, Inc. 4th Ed. 1999.

[5] Tabachnikov, S. *Billiards.* Panoramas Et Synthèses 1. Sociètè Mathèmatique de France. 1995.

[6] Vorobets, Ya.B., G.A. Gal'perin, and A.M. Stepin. *Periodic Billiard Trajectories in Polygons: Generating Mechanisms.* Russian Mathematical Surveys. Vol. 47, No. 3. 1992.

[7] Wilf, H. *generatingfunctionology.* Academic Press, Inc. 2nd Ed. 1994.