

# Periodic Orbits on a Triangular Air Hockey Table

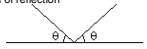
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## The Problem

- Examine periodic orbits on a triangular frictionless surface bounded by walls.
- A triangular air hockey table with a bouncing puck.
  - Triangular mirror room reflecting a laser.

## Assumptions

- A puck bounce follows the same rules as a reflection, that is, the angle of incidence equals the angle of reflection.



- A path terminates at the vertices.

## Definitions

A path followed is called an orbit.

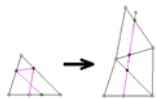
There are 3 types of orbits:

- A terminal orbit terminates at a vertex.
- An infinite orbit continues infinitely never retracing and never terminating at a vertex.
- A periodic orbit retraces itself after a finite number of bounces.

The number of bounces in a periodic orbit is its period.

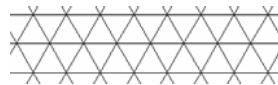
## A Technique from Geometry

A very useful technique in proving these theorems comes out of transformational geometry. Since the base angles in a bounce are equal, instead of tracing the path of the puck, we can keep the path a straight line and reflect the triangle about the lines that the path intersects. Any line intersected by the straight path in the expanded version is a side of the triangle from which the puck bounces.



## The Equilateral Case

We first focused on an equilateral triangle for its equiangular properties and its three lines of symmetry. Another benefit of this case is the ability to tessellate the plane with reflections. While it is possible to tessellate (tile) the plane with congruent triangles, those triangles are not necessarily reflections of one another along their shared border. In truth, only the equilateral triangle can tessellate the plane in this manner. These reflections are important for the reason expressed above: any series of bounces can be drawn as a straight line on this tessellation.



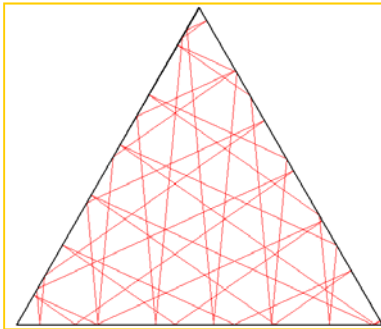
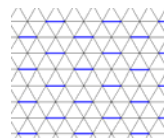
## Making the Periodic Orbit

By definition, a periodic orbit must repeat itself. Therefore the puck must do two things simultaneously:

- The puck must return to the initial point
- The bounce upon the return to the initial point must be equal to the initial angle.

These two factors translate quite easily into the tessellation. First, if we mark the

initial point before making the tessellation-producing reflections, we can trace the images of the initial through out the tessellation. By connecting the initial point to any of its images, the orbit will return to the initial point. However, the second condition is not necessarily fulfilled. To be sure the angle of incidence to the initial point is equal to the initial angle, we can use the fact that parallel lines cut by a transversal have equal corresponding angles. Thus, we can narrow our focus solely on the images of the initial point that lie on lines parallel to the initial side. These are called corresponding parallel images (or cpis). They are marked in blue below.



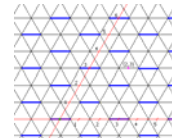
A Period 38 Orbit. Produced by Orbit Tracer

Any path (representable by a vector) drawn between two corresponding parallel images represents a periodic orbit, visible upon collapsing the reflections back into a single triangle.

In this new basis, the ideas of length and angle need to be adjusted to fit the new coordinates  $(x, y)$  a point would be given. With some basic trigonometry and algebra, we were able to come up with the following two formulae.

## A New Coordinate System

By superimposing axes on the tessellation, we create a new coordinate system. We set the origin to be the starting point, the x-axis as the horizontal, and the y-axis as parallel to the right-leaning diagonals (60° counter-clockwise from the horizontal).



Using linear algebra, we have created the change of basis matrices to and from this rhombic system.

Euclidean to Rhombic Rhombic to Euclidean

$$\begin{bmatrix} 1 & -\frac{\sqrt{3}}{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{bmatrix} \quad \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\text{Length } (x, y) = \sqrt{x^2 + xy + y^2}$$

$$\text{Angle } (x, y) = \tan^{-1} \left( \frac{y\sqrt{3}}{2x+y} \right)$$

While it is nice to have these formulae, they are useless unless we are able to tell which orbits  $(x, y)$  are periodic. This is a quite simple matter, with a relatively simple proof.

$(x, y)$  is a periodic orbit if and only if  $x = y \pmod{3}$

A sketch of the proof is as follows. First, one notices that  $(-1, 2)$  and  $(1, 1)$  are both periodic orbits. Furthermore, any other periodic orbits can be drawn as a linear combination of these two orbits, as per normal vector addition and scalar multiplication. Since  $-1 = 2 \pmod{3}$  and  $1 = 1 \pmod{3}$ , the x and y coordinates of any linear combination of the two would be congruent modulo 3.

One last formula stemming from this new basis is one to find the period of some periodic orbit  $(x, y)$ . The period is equal to the number of lines on the tessellation the vector crosses from beginning to end. By drawing a series of rhombi, each composed of pairs of equilateral triangles, along the path of the orbit, I have come up with the following formula:

$$\text{Period}(x, y) = \begin{cases} 2(x+y) & x \geq 0, y > 0 \\ 2y & x < 0, y > -x \\ -2x & x < 0, 0 < y < -x \end{cases}$$

You may note that the case when  $y < 0$  is unaccounted for in this formula. That is because the case when  $y < 0$  is unnecessary in our situation, because that means the initial angle was below the horizontal. Further narrowing our focus, we examine only the case when  $x > 0$ , that is, the initial angle is between 0° and 60° above the horizontal. Due to the geometry of the equilateral triangle, there must be at least one bounce angle between 0° and 60°, and this will be the angle we will take as the starting angle.

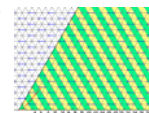
From this period formula we see there exist an infinite number of families of periodic orbits on the equilateral case. A family of periodic orbits is the set of all orbits that have the same vector  $(x, y)$ , although their starting points vary. Since the range of starting points is continuous over the interval, each family is infinite. Therefore there are an infinite number of infinite families of periodic orbits.

The most interesting results from the period formula shown above, you will notice that the period function is neither one-to-one nor onto. Every periodic orbit on the equilateral triangle has an even period, save for the special case of the period 3, the corresponding point of which does not lie on a cpi. As for the lack in injectivity, this leads to the question of how many periodic orbits  $(x, y)$  yield the same period? Through careful proof, we came upon the following:

There exist  $\frac{k-m}{3} + 1$  pairs of coordinates  $(x, y)$ , such that  $x = y \pmod{3}$  and  $\text{Period}(x, y) = 2h$ , where  $k =$  the greatest integer less than  $\frac{h}{2}$  and congruent to  $n \pmod{3}$

$m =$  the least nonnegative integer congruent to  $n \pmod{3}$  (i.e. 0, 1, or 2)

Graphically, all cpis that produce orbits of the same period lie in the same left leaning channel. The formula above counts the number of cpis in each channel.



## Open Question:

While we have proven that there exists a periodic orbit on every acute triangle, every right triangle, and every isosceles triangle, we have as of yet met with little success proving the existence of a periodic orbit on every scalene obtuse triangle. If we can find a proof of this, we have proven that a periodic orbit exists on every type of triangle.

## The Technology

At the start of our research we wanted to take a look at different situations in order to get a basic understanding of what happens in the triangular air hockey table. A basic understanding was necessary to direct our research and help us to focus where we were headed in this area. The first step in simulating the triangular air hockey table were various sketches from *The Geometer's SketchPad* made by Andrew Baxter.

Using sketchpad worked; however, the number of bounces had to be built into the sketch, and working through *Geometer's Sketchpad* was generally cumbersome. Therefore, we decided to program some of our own tools for studying this problem. *Orbit Tracer*, the first of these tools, is a program that simulates the bounces in the triangle. As inputs it takes the base angles of the triangle, the starting position of the puck, initial angle of departure, and the number of bounces to simulate. It iteratively uses the Law of Sines to find each subsequent bounce point and displays this information graphically using the Windows GDI. This program was also designed to let the user continuously modify the starting position and initial angle using the mouse. As a result, we could quickly observe how the puck bounced in a variety of situations, and how changing the starting position and angle affected the puck's path. Observations made using this program helped us come up with many of our conjectures that we later went on to prove. *Orbit Tracer* was used to create the central image of the period 38 orbit.

The second program we created for this research project is based on an observation made by Andrew Baxter. Baxter observed that an equilateral triangle can be tessellated in such a way as to straighten out the path of the puck. Finding an orbit then becomes connecting parallel corresponding sides. At first we drew the paths on paper with a triangular tessellation printed on it. However, having this paper available and keeping track of corresponding sides became a problem, so we created a program to do it for us. The program draws a resizable tessellation on the screen and highlights the corresponding parallel sides. The user can then click on one of the highlighted sides and the program draws the orbit path between those sides. This program then uses the *Orbit Tracer* program to show what the path looks like inside one triangle. The initial angle of the path is determined by which corresponding side the user chooses, and the initial position can be chosen using the mouse. This program will also display information about the path based on formulae created by Baxter. The tessellation idea and this program were instrumental in studying the equilateral case.

These programs were very helpful in gaining a basic understanding of what happens in a triangular air hockey table. However, since they only provides specific examples, they could not be used to directly prove our conjectures. Also, as our research has progressed, we have gotten away from basic observations and more into the mathematical side of the problem. As a result we are not relying on these programs as much as we once did. However, they are experiencing a resurgence as we tackle the final leg of the problem: proving the existence of a periodic orbit on the scalene obtuse triangle.