2-In/3-Out A_∞-Bialgebras

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Abstract

A *bialgebra* is a vector space H equipped with a multiplication $\mu : H \otimes H \to H$ and a compatible comultiplication $\Delta : H \to H \otimes H$. A 2-in/3-out A_∞-*bialgebra* is a vector space H equipped with three compatible operations: a multiplication μ, a comultiplication Δ, and an operation $\omega : H \otimes H \to H \otimes H \otimes H$. We present an original example and verify that it satifies the compatibility axioms of a 2-in/3-out A_∞-bialgebra. This is the first known example of this particular structure. We are interested in A_∞-bialgebras because every loop space has an associated A_∞-bialgebra. Whether or not every A_∞-bialgebra corresponds to some loop space is an open question.

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Contents

1 Introduction

A 2-in/3-out A_∞-bialgebra is an algebraic structure with three operations that satisfy nine compatibility relations. This thesis introduces the first known example of such a structure. The need for examples arose from the work of my thesis advisor, Dr. Ronald Umble and his collaborator Dr. Samson Saneblidze, who proved that every loop space has an A_∞-bialgebra model. Whether or not every A_∞-bialgebra corresponds to some loop space is an open question. Examples of A_∞-bialgebras would be helpful in examining the converse of Saneblidze and Umble's theorem.

We begin with a series of definitions that lead up to the definition of a 2-in/3-out A_∞-bialgebra.

2 Key Definitions and Background Information

Definition 1 *A* Field $(F,+,\cdot\ )$ *is a set* F *with addition* $+$ *and multiplication* $\cdot$ *such that*

1. $(F,+\ )$ *is an abelian group.*
2. $(F-0,\cdot\ )$ *is an abelian group.*
3. *Multiplication distributes over addition.*

Definition 2 *Fix a field* F*. The elements of* F *are called scalars and the multiplicative identity element is denoted by* 1_F*. An* F*-Vector Space is an abelian group* $(V,+)$ *together with a scalar multiplication denoted by juxtaposition such that*

1. $\forall a\in F$ *and* $v\in V$, $av\in V$.
2. $\forall a\in F$ *and* $v,w\in V$, $a(v+w)=av+aw$.
3. $\forall a,b\in F$ *and* $v\in V$, $(a+b)v=av+bv$.
4. $\forall a,b\in F$ *and* $v\in V$, $(ab)v=a(bv)$.

5. $\forall v \in V$, $1_F v = v$.

In the discussion that follows, all vector spaces are defined over F.

Definition 3 *Let V and W be finite vector spaces with respective bases $\{v_1, v_2, \dots v_n\}$ and $\{w_1, w_2, \dots w_m\}$. The tensor product of V and W is the vector space $V \otimes W$ with basis containing elements of the form $v_i \otimes w_j$, for $i \leq n$ and $j \leq m$, such that*

1. *$\forall a \in F$ and $v \otimes w \in V \otimes W$, $a(v \otimes w) = av \otimes w = v \otimes aw$.*
2. *$\forall u, v \in V$ and $w \in W$, $(u + v) \otimes w = u \otimes w + v \otimes w$.*
3. *$\forall v \in V$ and $u, w \in W$, $v \otimes (u + w) = v \otimes u + v \otimes w$.*

Definition 4 *Let V be a vector space. A* differential *on a vector space V is a linear map $d : V \to V$ such that $d \circ d = 0$. A* graded vector space *is a vector space V that can be decomposed as a direct sum of vector subspaces $\{V_i\}_{i \geq 0}$, i.e.,*

$$V = \bigoplus_{i=0}^{\infty} V_i.$$

An element $v \in V_i$ has degree *i and the symbol $|v|$ denotes the degree of v. Let V and W be graded vector spaces. A linear map $f : A \to B$ has degree p if $f(A_i) \subset B_{i+p}$ for each i. A* differential graded vector space *is a graded vector space equipped with a differential d of degree -1, i.e., $d : V_i \to V_{i-1}$ for all i.*

Let $1 : A \to A$ be the identity map.

Definition 5 *An* algebra *over F is a vector space A equipped with an associative multiplication $\mu : A \otimes A \to A$, i.e.,*

$$\mu (\mu \otimes 1) = \mu (1 \otimes \mu) : A^{\otimes 3} \to A.$$

An algebra A is unital *if there is an element $\mathbf{1} \in A$ such that $\mathbf{1}a = a\mathbf{1} = a$ for all $a \in A$.*

Definition 6 *A* graded algebra *A is an algebra whose multiplication μ satisfies*

$$\mu\left(A_i \otimes A_j\right) \subseteq A_{i+j}.$$

Definition 7 *A* differential graded algbera *A is a both a graded algebra and a differential graded vector space whose differential d is a graded* derivation , *i.e.,*

$$\begin{aligned} d\,(ab) &= (da)\,b + (-1)^{|a|} a\,(db) \qquad (1) \\ d\mu &= \mu\,(d \otimes 1 + 1 \otimes d) \end{aligned}$$

The sign in equation (1) arises from Mac Lane's sign commutation rule [1]: *Whenever the positions of two graded objects, a and b, are interchanged, afix the sign* $(-1)^{|a||b|}$. In equation (1), the positions of a and b are interchanged as the expression on the righthand side is applied to $a \otimes b$.

Let (A, μ_A) and (B, μ_B) be graded algebras. A map $f : A \to B$, of degree 0, is an algebra map if

$$f\mu_A = \mu_B(f \otimes f)$$

Definition 8 *A* coalgebra *over F is a vector space C equipped with a coassociative comultiplication $\Delta : C \to C \otimes C$, i.e.,*

$$(\Delta \otimes \mathrm{Id})\,\Delta = (\mathrm{Id} \otimes \Delta)\,\Delta.$$

Definition 9 *A* graded coalgebra *is a coalgebra (C, Δ) such that*

$$\Delta(C_p) \subseteq \bigoplus_{r+s=p} C_r \otimes C_s\ .$$

Definition 10 *A* differential graded coalgebra *is both a graded coalgebra C and a differential graded vector space whose differential d is a coderivation, i.e.,*

$$\Delta d = (d \otimes Id + Id \otimes d)\,\Delta$$

A tensor product is a vector space, thus it is possible to tensor an already tensored space. For example, $(V^{\otimes p})^{\otimes q}$ denotes $V^{\otimes p}$ tensored with itself q times. For this paper we will often use the symbol $|$ as an alternative sign for $\otimes$ to help distinguish elements in $V^{\otimes p}$.

Definition 11 *Let V be a graded vector space. Define $\sigma_{p,q} : (V^{\otimes p})^{\otimes q} \to (V^{\otimes q})^{\otimes p}$ to be the canonical permutation of tensor factors. For example,*

$$\sigma_{3,2}[a_1|b_1|c_1 \otimes a_2|b_2|c_2] = (-1)^{|b_1||a_2|+|c_1||a_2|+|c_1||b_2|}(a_1|a_2 \otimes b_1|b_2 \otimes c_1|c_2),$$

where the sign is given by the sign commutation rule.

Definition 12 *A* bialgebra *is a graded vector space H, equipped with a multiplication $\mu : H \otimes H \to H$ and a comultiplication $\Delta : H \to H \otimes H$ that satisfy the Hopf Relation:*

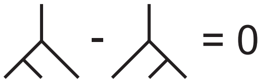
$$\Delta\mu = (\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta)$$

Definition 13 *Let H be a graded vector space equipped with three operations:*

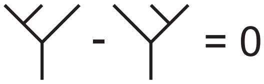
- $\mu : H \otimes H \to H$.
- $\Delta : H \to H \otimes H$.
- $\omega : H \otimes H \to H \otimes H \otimes H$.

Then (H, μ, Δ, ω) ia a 2-in/3-out A_∞-bialgbera if the following relations are satisfied:

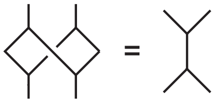
1. $\mu(\mu \otimes 1 - 1 \otimes \mu) = 0$ *(μ is associative).*



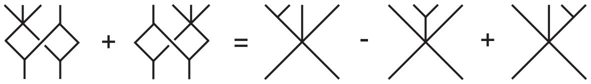
2. $(\Delta \otimes 1 - 1 \otimes \Delta)\Delta = 0$ *(Δ is coassociative).*



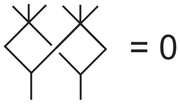
3. $(\mu \otimes \mu)\, \sigma_{2,2}\, (\Delta \otimes \Delta) = \Delta\mu$ *(H is a bialgebra).*



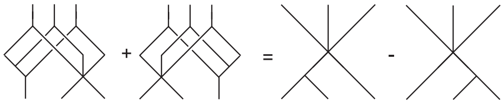
4. $(\omega \otimes \mu + \mu \otimes \omega)\, \sigma_{2,2}\, (\Delta \otimes \Delta) = (\Delta \otimes 1^{\otimes 2} - 1 \otimes \Delta \otimes 1 + 1^{\otimes 2} \otimes \Delta)\, \omega.$



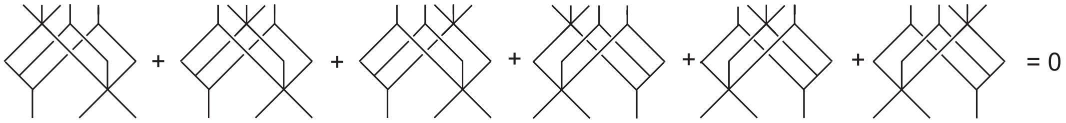
5. $(\omega \otimes \omega)\, \sigma_{2,2}\, (\Delta \otimes \Delta) = 0.$



6. $\mu^{\otimes 3} \sigma_{3,2}\, (f \otimes \omega + \omega \otimes g) = \omega\, (\mu \otimes 1 - 1 \otimes \mu)\, .$



7. $(\mu^{\otimes 2} \otimes \omega + \mu \otimes \omega \otimes \mu + \omega \otimes \mu^{\otimes 2})\, \sigma_{3,2}\, (f \otimes \omega + \omega \otimes g) = 0.$



8. $(\mu \otimes \omega^{\otimes 2} + \omega \otimes \mu \otimes \omega + \omega^{\otimes 2} \otimes \mu)\, \sigma_{3,2}\, (f \otimes \omega + \omega \otimes g) = 0.$

   (Relation 8 is similar to Relation 7 except with two ω operations permuted along the top of each component.)

9. $\omega^{\otimes 3} \sigma_{3,2}\, (f \otimes \omega + \omega \otimes g) = 0.$

   (Relation 9 is similar to Relation 6 except with three ω operations along the top of each component and right-hand side 0.*)*

A 2-in/3-out A_∞-bialgebra is a special case of more general A_∞-bialgebras introduced by Saneblidze and Umble in [2].

Definition 14 *The* tensor vector space *generated by H, is the graded vector space*

$$TH = F \oplus \bigoplus_{i \geq 1} H^{\otimes i} \ ,$$

where $|a_1 \otimes \cdots \otimes a_n| = \sum_{i=1}^{n} |a_i|$ and $|x| = 0$, $\forall x \in F$. An element of TH is a linear combination of tensors of varying lengths. For example,

$$x_1|x_2 + 2x_3|x_4|x_5$$

Definition 15 *The* tensor algebra *of H, denoted by T^aH is the tensor vector space TH with multiplication given by juxtaposition.*

Definition 16 *The* tensor coalgebra *of H, denoted by T^cH, is the tensor vector space TH with comultiplication given by $\Delta^c(a) = a(1 \otimes 1)$, for $a \in F$, and $\Delta^c(a_1|\ldots|a_n) = \sum_{i=1}^{n-1} a_1|\ldots|a_i \otimes a_{i+1}|\ldots|a_n$, for $(a_1|\ldots|a_n) \in H^{\otimes i}$*

The relations in a 2-in/3-out A_∞-bialgebra arise in the following way:

1. Extend $\Delta : H \to H^{\otimes 2}$ as a graded derivation $\widetilde{\Delta} : T^aH \to T^aH$ with respect to multiplication in T^aH. Then

$$\widetilde{\Delta} = \Delta + (\Delta \otimes 1 - 1 \otimes \Delta) + \left(\Delta \otimes 1^{\otimes 2} - 1 \otimes \Delta \otimes 1 + 1^{\otimes 2} \otimes \Delta\right) + \cdots ,$$

   where the signs are introduced so that $\widetilde{\Delta} \circ \widetilde{\Delta} = 0$ when Δ is coassociative. When $\widetilde{\Delta}$ is applied to $a_1|a_2|\cdots|a_n$, all components of $\widetilde{\Delta}$ with tensor length other than n are zero. Thus

$$\widetilde{\Delta}\left(a_1|a_2|\cdots|a_n\right) = \Delta a_1|a_2|\cdots|a_n - a_1|\Delta a_2|\cdots|a_n + \cdots + (-1)^{n-1} a_1|a_2|\cdots|\Delta a_n.$$

2. Extend $\mu : H^{\otimes 2} \to H$ as a graded coderivation $\widetilde{\mu} : T^c H \to T^c H$ with respect to the comultiplication Δ^c in $T^c H$. Then

$$\widetilde{\mu} = \mu + (\mu \otimes 1 - 1 \otimes \mu) + \left(\mu \otimes 1^{\otimes 2} - 1 \otimes \mu \otimes 1 + 1^{\otimes 2} \otimes \mu\right) + \cdots ,$$

   where the signs are introduced so that $\widetilde{\mu} \circ \widetilde{\mu} = 0$ when μ is associative. When $\widetilde{\mu}$ is applied to $a_1|a_2|\cdots|a_n$, all components of $\widetilde{\mu}$ with tensor length other than $n-1$ are zero. Thus

$$\begin{aligned} \widetilde{\mu}\,(a_1|a_2|\cdots|a_n) &= \mu\,(a_1|a_2)\,|\cdots|a_n - a_1|\mu\,(a_2|a_3)\,|\cdots|a_n + \cdots \\ &\quad + (-1)^{n-2}\, a_1|a_2|\cdots|\mu\,(a_{n-1}|a_n)\,. \end{aligned}$$

3. Extend $\Delta : H \to H^{\otimes 2}$ as a map $\overline{\Delta} : T^a H \to T^a\left(H^{\otimes 2}\right)$ of tensor algebras. Then

$$\overline{\Delta} = \Delta + \Delta \otimes \Delta + \Delta \otimes \Delta \otimes \Delta + \cdots .$$

   When $\overline{\Delta}$ is applied to $a_1|a_2|\cdots|a_n$, all components of $\overline{\Delta}$ with tensor length other than n are zero. Thus $\overline{\Delta}\,(a_1|a_2|\cdots|a_n) = \Delta a_1|\Delta a_2|\cdots|\Delta a_n$.

4. Extend $\mu + \omega : H^{\otimes 2} \to T^c(H)$ as a map of tensor coalgebras $\overline{\mu + \omega} : T^c\left(H^{\otimes 2}\right) \to T^c(H)$. Then

$$\begin{aligned} \overline{\mu + \omega} &= (\mu + \omega) + (\mu + \omega)^{\otimes 2} + (\mu + \omega)^{\otimes 3} + \cdots \\ &= (\mu + \omega) + \left(\mu^{\otimes 2} + \mu \otimes \omega + \omega \otimes \mu + \omega^{\otimes 2}\right) \\ &\quad + \left(\mu^{\otimes 3} + \mu^{\otimes 2} \otimes \omega + \mu \otimes \omega \otimes \mu + \omega \otimes \mu^{\otimes 2}\right. \\ &\quad \left. + \mu \otimes \omega^{\otimes 2} + \omega \otimes \mu \otimes \omega + \omega^{\otimes 2} \otimes \mu + \omega^{\otimes 3}\right) + \cdots \end{aligned}$$

5. Note that $f = (\Delta \otimes 1)\,\Delta$, $g = (1 \otimes \Delta)\,\Delta : H \to H^{\otimes 3}$ are algebra maps and extend ω as an (f,g)-derivation $\overline{\omega} : T^a(H) \to T^a\left(H^{\otimes 3}\right)$. Then

$$\overline{\omega} = f \otimes \omega + \omega \otimes g + \cdots .$$

The biderivative d_ω is the sum of all maps in (1)-(5) above. Thus the structure relations in the related 2-in/3-out A_∞-bialgebra are the homogeneous components of $d_\omega \circ d_\omega = 0$.

3 An example of a 2-in/3-out A_∞-bialgebra structure

Let H be a graded vector space with basis $\{1, y\}$, where $|1| = 0$ and $|y| = 2$, and define

$$\mu(a|b) = \begin{cases} 1, & \text{if } a|b = 1|1 \\ y, & \text{if } a|b \in \{1|y,\ y|1\} \\ 0, & \text{otherwise} \end{cases} \qquad \Delta(a) = \begin{cases} 1|1, & \text{if } a = 1 \\ 1|y + y|1, & \text{if } a = y \end{cases}$$

$$\omega(a|b) = \begin{cases} y|1|1 + 1|1|y, & \text{if } a|b = y|y \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 17 *$(H, \mu, \Delta,)$ is a bialgebra.*

To prove that (H, μ, Δ) is a bialgebra, we must verify that multiplication is associative, that comultiplication is coassociative, and that the Hopf relation is satisfied. These three conditions appear as relations (1)-(3) in Definition 11. We Check all possibilities in each case.

Relation 1 (μ is associative): $\mu(\mu \otimes 1 - 1 \otimes \mu) = 0$.

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(y|y|y) = \mu(y^2|y - y|y^2) = y^3 - y^3 = 0.$$

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(y|y|1) = \mu(y^2|1 - y|y) = (y^2) - (y^2) = 0.$$

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(y|1|y) = \mu(y|y - y|y) = 0.$$

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(y|1|1) = \mu(y|1 - y|1) = 0.$$

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(1|y|y) = \mu(y|y - 1|y^2) = y^2 - y^2 = 0.$$

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(1|y|1) = \mu(y|1 - 1|y) = y - y = 0.$$

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(1|1|y) = \mu(1|y - 1|y) = 0.$$

$$\mu(\mu \otimes 1 - 1 \otimes \mu)(1|1|1) = \mu(1|1 - 1|1) = 0.$$

Relation 2 (Δ is coassociative): $(\Delta \otimes 1 - 1 \otimes \Delta)\Delta = 0$.

$$(\Delta \otimes 1 - 1 \otimes \Delta)\Delta(y) = (\Delta \otimes 1 - 1 \otimes \Delta)(y|1 + 1|y)$$

$$= (y|1 + 1|y) \otimes 1 - y|1|1 + 1|1|y - 1 \otimes (y|1 + 1|y)$$

$$= y|1|1 + 1|y|1 - y|1|1 + 1|1|y - 1|y|1 - 1|1|y = 0.$$

$$(\Delta \otimes 1 - 1 \otimes \Delta)\Delta(1) = 1|1|1 - 1|1|1 = 0.$$

Relation 3 (The Hopf Relation): $(\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta) = \Delta(\mu)$.

$$(\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta)(y|y) = (\mu \otimes \mu)\sigma_{2,2}[(y|1 + 1|y) \otimes (y|1 + 1|y)]$$

$$= (\mu \otimes \mu)(y|y|1|1 + 1|y|y|1 + y|1|1|y + 1|1|y|y)$$

$$= y^2|1 + 2(y|y) + 1|y^2 = \Delta(y^2) = \Delta(\mu)(y|y).$$

$$(\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta)(y|1) = (\mu \otimes \mu)\sigma_{2,2}[(y|1 + 1|y) \otimes (1|1)]$$

$$= (\mu \otimes \mu)(y|1|1|1 + 1|1|y|1) = y|1 + 1|y = \Delta(y) = \Delta(\mu)(y|1).$$

$$(\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta)(1|y) = (\mu \otimes \mu)\sigma_{2,2}[(1|1) \otimes (y|1 + 1|y)]$$

$$= (\mu \otimes \mu)(1|y|1|1 + 1|1|1|y) = y|1 + 1|y = \Delta(y) = \Delta(\mu)(1|y).$$

$$(\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta)(1|1) = (\mu \otimes \mu)\sigma_{2,2}[(1|1) \otimes (1|1)]$$

$$= (\mu \otimes \mu)(1|1|1|1) = 1|1 = \Delta(1) = \Delta(\mu)(1|1).$$

Theorem 18 *(H, μ, Δ, ω) is a 2-in/3-out A_∞-bialgebra.*

We must verify relations (4)-(9) from Definition 11. Again, we check all possibilities in each case.

Relation 4: $(\omega \otimes \mu + \mu \otimes \omega)\sigma_{2,2}(\Delta \otimes \Delta) = (\Delta \otimes 1^{\otimes 2} - 1 \otimes \Delta \otimes 1 + 1^{\otimes 2} \otimes \Delta)\omega$.

$$(\omega \otimes \mu + \mu \otimes \omega)\sigma_{2,2}(\Delta \otimes \Delta)(y|y)$$

$$= (\omega \otimes \mu + \mu \otimes \omega)(y|y|1|1 + y|1|1|y + 1|y|y|1 + 1|1|y|y$$

$$= (y|1|1 + 1|1|y) \otimes 1 + 1 \otimes (y|1|1 + 1|1|y)$$

$$= y|1|1|1 + 1|1|y|1 + 1|y|1|1 + 1|1|1|y$$

$$= y|1|1|1 + 1|y|1|1 - y|1|1|1 + y|1|1|1 + 1|1|1|y - 1|1|1|y + 1|1|y|1 + 1|1|1|y$$

$$= (y|1 + 1|y) \otimes (1|1) - y \otimes (1|1) \otimes 1 + (y|1) \otimes (1|1)$$
$$+(1|1) \otimes (1|y) - 1 \otimes (1|1) \otimes y + (1|1) \otimes (y|1 + 1|y)$$

$$= (\Delta \otimes 1^{\otimes 2} - 1 \otimes \Delta \otimes 1 + 1^{\otimes 2} \otimes \Delta)\omega(y|y).$$

$$(\omega \otimes \mu + \mu \otimes \omega)\sigma_{2,2}(\Delta \otimes \Delta)(y|1) = (\omega \otimes \mu + \mu \otimes \omega)\sigma_{2,2}[(y|1 + 1|y) \otimes (1|1)] = 0$$

$$= (\Delta \otimes 1^{\otimes 2} - 1 \otimes \Delta \otimes 1 + 1^{\otimes 2} \otimes \Delta)\omega(y|1).$$

$$(\omega\otimes\mu+\mu\otimes\omega)\sigma_{2,2}(\Delta\otimes\Delta)(1|y)=(\omega\otimes\mu+\mu\otimes\omega)\sigma_{2,2}[(1|1)\otimes(y|1+1|y)]=0$$

$$=(\Delta\otimes 1^{\otimes 2}-1\otimes\Delta\otimes 1+1^{\otimes 2}\otimes\Delta)\omega(y|1).$$

$$(\omega\otimes\mu+\mu\otimes\omega)\sigma_{2,2}(\Delta\otimes\Delta)(1|1)=(\omega\otimes\mu+\mu\otimes\omega)\sigma_{2,2}[(1|1)\otimes(1|1)]=0$$

$$=(\Delta\otimes 1^{\otimes 2}-1\otimes\Delta\otimes 1+1^{\otimes 2}\otimes\Delta)\omega(1|1).$$

Relation 5: $(\omega\otimes\omega)\sigma_{2,2}(\Delta\otimes\Delta)=0$.

$$(\omega\otimes\omega)\sigma_{2,2}(\Delta\otimes\Delta)(y|y)=(\omega\otimes\omega)\sigma_{2,2}[(y|1+1|y)\otimes(y|1+1|y)]=0.$$

$$(\omega\otimes\omega)\sigma_{2,2}(\Delta\otimes\Delta)(y|1)=(\omega\otimes\omega)\sigma_{2,2}[(y|1+1|y)\otimes(1|1)]=0.$$

$$(\omega\otimes\omega)\sigma_{2,2}(\Delta\otimes\Delta)(1|y)=(\omega\otimes\omega)\sigma_{2,2}[(1|1)\otimes(y|1+1|y)]=0.$$

$$(\omega\otimes\omega)\sigma_{2,2}(\Delta\otimes\Delta)(1|y)=(\omega\otimes\omega)\sigma_{2,2}[(1|1)\otimes(1|1)]=0.$$

Relations (6)-(9) contain the factor $\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta]$. We will now calculate this common factor, then apply the result to relations (6)-(9).

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](y|y|y)$$

$$=\sigma_{3,2}[(y|1|1+1|y|1+1|1|y)\otimes(y|1|1+1|1|y)$$

$$-(y|1|1+1|1|y)\otimes(y|1|1+1|y|1+1|1|y)]$$

$$=y|y|1|1|1|1+y|1|1|1|1|y+1|y|y|1|1|1+1|1|y|1|1|y$$

$$+1|y|1|1|y|1+1|1|1|1|y|y-y|y|1|1|1|1-y|1|1|y|1|1$$

$$-y|1|1|1|1|y-1|y|1|1|y|1-1|1|1|y|y|1-1|1|1|1|y|y$$

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](y|y|1)=$$

$$=\sigma_{3,2}[0-(y|1|1+1|1|y)\otimes(1|1|1)]=-y|1|1|1|1|1-1|1|1|1|y|1.$$

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](y|1|y)$$

$$=\sigma_{3,2}[(\Delta\otimes 1)\Delta(y)\otimes 0-0\otimes(1\otimes\Delta)\Delta(y)]=0.$$

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](y|1|1)$$

$$=\sigma_{3,2}[(\Delta\otimes 1)\Delta(y)\otimes 0-0\otimes(1\otimes\Delta)\Delta(1)]=0.$$

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](1|y|y)$$

$$=\sigma_{3,2}[(1|1|1)\otimes(y|1|1+1|1|y)-0]=1|y|1|1|1|1+1|1|1|1|1|y.$$

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](1|y|1)$$

$$=\sigma_{3,2}[(\Delta\otimes 1)\Delta(1)\otimes 0-0\otimes(1\otimes\Delta)\Delta(1)]=0.$$

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](1|1|y)$$

$$=\sigma_{3,2}[(\Delta\otimes 1)\Delta(1)\otimes 0-0\otimes(1\otimes\Delta)\Delta(y)]=0.$$

$$\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta](1|1|1)$$

$$=\sigma_{3,2}[(\Delta\otimes 1)\Delta(1)\otimes 0-0\otimes(1\otimes\Delta)\Delta(1)]=0.$$

Let $A=\mu^{\otimes 3}\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega-\omega\otimes(1\otimes\Delta)\Delta]$.

Relation 6: $A=\omega(\mu\otimes 1-1\otimes\mu)$.

$$A(y|y|y)=\mu^{\otimes 3}((1|y)\otimes(y|1)\otimes(1|1)+(1|1)\otimes(y|1)\otimes(1|y)$$

$$-(y|1)\otimes(1|y)\otimes(1|1)-(1|1)\otimes(1|y)\otimes(y|1))$$

$$=y|y|1+1|y|y-y|y|1-1|y|y=0=\omega(y^2|y-y|y^2)=\omega(\mu\otimes 1-1\otimes\mu)(y|y|y).$$

$$A(y|y|1)=\mu^{\otimes 3}(-(y|1)\otimes(1|1)\otimes(1|1)-(1|1)\otimes(1|1)\otimes(y|1))$$

$$=-y|1|1-1|1|y=-\omega(y|y)=\omega(y^2|1-y|y)=\omega(\mu\otimes 1-1\otimes\mu)(y|y|1).$$

$$A(y|1|y)=\mu^{\otimes 3}(0)=0=\omega(0)=\omega(y|y-y|y)=\omega(\mu\otimes 1-1\otimes\mu)(y|1|y).$$

$$A(y|1|1)=\mu^{\otimes 3}(0)=0=\omega(y|1-y|1)=\omega(\mu\otimes 1-1\otimes\mu)(y|1|1).$$

$$A(1|y|y)=\mu^{\otimes 3}((1|y)\otimes(1|1)\otimes(1|1)+(1|1)\otimes(1|1)\otimes(1|y))$$

$$=y|1|1+1|1|y=\omega(y|y)=\omega(y|y-1|y^2)=\omega(\mu\otimes 1-1\otimes\mu)(1|y|y).$$

$$A(1|y|1)=\mu^{\otimes 3}(0)=0=\omega(y|1-1|y)=\omega(\mu\otimes 1-1\otimes\mu)(1|y|1).$$

$$A(1|1|y) = \mu^{\otimes 3}(0) = 0 = \omega(1|y - 1|y) = \omega(\mu \otimes 1 - 1 \otimes \mu)(1|1|y).$$

$$A(1|1|1) = \mu^{\otimes 3}(0) = 0 = \omega(1|1 - 1|1) = \omega(\mu \otimes 1 - 1 \otimes \mu)(1|1|1).$$

Let $B = (\omega \otimes \mu^{\otimes 2} + \mu \otimes \omega \otimes \mu + \mu^{\otimes 2} \otimes \omega)\sigma_{3,2}[(\Delta \otimes 1)\Delta \otimes \omega - \omega \otimes (1 \otimes \Delta)\Delta]$.

Relation 7: $B = 0$.

$$B(y|y|y) = (\omega \otimes \mu^{\otimes 2} + \mu \otimes \omega \otimes \mu + \mu^{\otimes 2} \otimes \omega)\,(1|y \otimes y|1 \otimes 1|1$$

$$+1|1 \otimes y|1 \otimes 1|y - y|1 \otimes 1|y \otimes 1|1 - 1|1 \otimes 1|y \otimes y|1) = 0.$$

$$B(y|y|1) = (\omega \otimes \mu^{\otimes 2} + \mu \otimes \omega \otimes \mu + \mu^{\otimes 2} \otimes \omega)(-y|1 \otimes 1|1 \otimes 1|1 - 1|1 \otimes 1|1 \otimes y|1) = 0.$$

$$B(1|y|y) = (\omega \otimes \mu^{\otimes 2} + \mu \otimes \omega \otimes \mu + \mu^{\otimes 2} \otimes \omega)(1|y \otimes 1|1 \otimes 1|1 + 1|1 \otimes 1|1 \otimes 1|y) = 0.$$

$$B(y|1|y) = B(y|1|1) = B(1|y|1) = B(1|1|y) = B(1|1|1)$$

$$= (\omega \otimes \mu^{\otimes 2} + \mu \otimes \omega \otimes \mu + \mu^{\otimes 2} \otimes \omega)(0) = 0.$$

Let $C = (\omega \otimes \omega \otimes \mu + \omega \otimes \mu \otimes \omega + \mu \otimes \omega \otimes \omega)\sigma_{3,2}[(\Delta \otimes 1)\Delta \otimes \omega - \omega \otimes (1 \otimes \Delta)\Delta]$.

Relation 8: $C = 0$.

$$C(y|y|y) = (\omega \otimes \omega \otimes \mu + \omega \otimes \mu \otimes \omega + \mu \otimes \omega \otimes \omega)(1|y \otimes y|1 \otimes 1|1$$

$$+1|1 \otimes y|1 \otimes 1|y - y|1 \otimes 1|y \otimes 1|1 - 1|1 \otimes 1|y \otimes y|1) = 0.$$

$$C(y|y|1) = (\omega \otimes \omega \otimes \mu + \omega \otimes \mu \otimes \omega + \mu \otimes \omega \otimes \omega)(-y|1 \otimes 1|1 \otimes 1|1 - 1|1 \otimes 1|1 \otimes y|1) = 0.$$

$$C(y|1|y) = (\omega\otimes\omega\otimes\mu + \omega\otimes\mu\otimes\omega + \mu\otimes\omega\otimes\omega)(0) = 0.$$

$$C(y|1|1) = (\omega\otimes\omega\otimes\mu + \omega\otimes\mu\otimes\omega + \mu\otimes\omega\otimes\omega)(0) = 0.$$

$$C(1|y|y) = (\omega\otimes\omega\otimes\mu + \omega\otimes\mu\otimes\omega + \mu\otimes\omega\otimes\omega)(1|y\otimes 1|1\otimes 1|1 + 1|1\otimes 1|1\otimes 1|y) = 0.$$

$$C(1|y|1) = (\omega\otimes\omega\otimes\mu + \omega\otimes\mu\otimes\omega + \mu\otimes\omega\otimes\omega)(0) = 0.$$

$$C(1|1|y) = (\omega\otimes\omega\otimes\mu + \omega\otimes\mu\otimes\omega + \mu\otimes\omega\otimes\omega)(0) = 0.$$

$$C(1|1|1) = (\omega\otimes\omega\otimes\mu + \omega\otimes\mu\otimes\omega + \mu\otimes\omega\otimes\omega)(0) = 0.$$

Let $D = (\omega^{\otimes 3})\sigma_{3,2}[(\Delta\otimes 1)\Delta\otimes\omega - \omega\otimes(1\otimes\Delta)\Delta]$.

Relation 9: $D = 0$.

$$D(y|y|y) = (\omega^{\otimes 3})(1|y\otimes y|1\otimes 1|1 + 1|1\otimes y|1\otimes 1|y - y|1\otimes 1|y\otimes 1|1 - 1|1\otimes 1|y\otimes y|1) = 0.$$

$$D(y|y|1) = (\omega^{\otimes 3})(-y|1\otimes 1|1\otimes 1|1 - 1|1\otimes 1|1\otimes y|1) = 0.$$

$$D(y|1|y) = (\omega^{\otimes 3})(0) = 0.$$

$$D(y|1|1) = (\omega^{\otimes 3})(0) = 0.$$

$$D(1|y|y) = (\omega^{\otimes 3})(1|y\otimes 1|1\otimes 1|1 + 1|1\otimes 1|1\otimes 1|y) = 0.$$

$$D(1|y|1) = (\omega^{\otimes 3})(0) = 0.$$

$D(1|1|y) = (\omega^{\otimes 3})(0) = 0.$

$D(1|1|1) = (\omega^{\otimes 3})(0) = 0.$

4 Conclusion

By exhaustion, we have verified that the graded vector space H generated by $\{1, y\}$ is an example of a 2-in/3-out A_∞-bialgebra with operation $\omega(y|y) = y|1|1 + 1|1|y$. It is possible that other 2-in/3-out A_∞-bialgebras exist, but this is a question for further study. Some other operations for ω were considered, but they fail to satisfy certain compatibility relations. For example, the reader can verify $\omega(y|y) = y|1|1 - 1|y|1 + 1|1|y$ fails relation 4.

References

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