The General Brachistochrone Problem

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Introduction to the Problem

Consider the following problem: Given two points *A* and *B* on some frictionless surface *S*, what curve is traced on *S* by a particle that starts at *A* and falls to *B* in the shortest time? Newton showed that if the surface is a plane and if the particle is moving in a uniform gravitational field the solution is a cycloid, the curve traced by the rim of a rolling circle. In our problem, the particle is falling on an arbitrary surface and the gravitational force is not necessarily uniform.

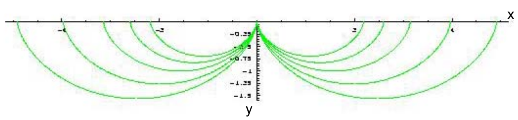

Fig. 1 Newton's Cycloid Solutions

General Theorem

To solve this problem in general we begin with conservation of Newtonian mechanical energy:

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2+V(u,v)=V(A),$$

where *u* and *v* are the generalized coordinates on *S* and *V* is the gravitational potential energy of the falling particle. Therefore, the total time *T* is given by:

$$T=\pm\sqrt{\tfrac{m}{2}}\int_A^B\sqrt{\tfrac{1}{V(A)-V}}\,ds=\int_A^B F[u,v,v']du.$$

To find the curve that minimizes *T* we need to solve the Euler-Lagrange equation:

$$\frac{d}{du}\frac{\partial F}{\partial v'}-\frac{\partial F}{\partial v}=0.$$

In general, solving the Euler-Lagrange equation involves solving a non-linear second order differential equation. We were able to determine conditions on *F* that allowed us to reduce the equation to a simple first order separable equation.

Theorem:

Let $\mathbf{x}(u,v)$ be a regular, injective parameterization of a surface *S* with metric $ds^2=E(u,v)du^2+G(u,v)dv^2$ lying in a gravitational field with potential energy $V(u,v)$. If *V, E,* and *G* are independent of *v* then the solution to the Brachistochrone problem on *S* is given by the curve $\mathbf{x}(u,v(u))$ where

$$v(u)=\pm\int_A^v\sqrt{\frac{C^2E(w)[V(A)-V(w)]}{G(w)\left[G(w)-C^2(V(A)-V(w))\right]}}\,dw.$$

The value of *C* is chosen so that the solution curve passes through the final point *B*.

Surfaces of Revolution

One important application of this theorem is for particles falling on a surface of revolution about the *z*-axis. In general a parameterization of a surface of revolution is given by:

$$\mathbf{x}(u,v)=\big(h(u)\cos v,h(u)\sin v,g(u)\big),$$

with metric

$$ds^2=\left(h'(u)^2+g'(u)^2\right)du^2+h(u)^2dv^2.$$

Therefore, we can apply the theorem to surfaces of revolution in uniform gravitational fields in which the axis of revolution is parallel to the direction of the field.

For example, figure 2 illustrates several solution curves on the right circular cone parameterized by

$$\mathbf{x}(u,v)=\big(u\cos v,u\sin v,u\big)$$

and lying in a gravitational field with $V=u$. Each of these curves is uniquely determined by *C* and in some sense represents a family of curves since the falling particle could terminate anywhere on the curve.

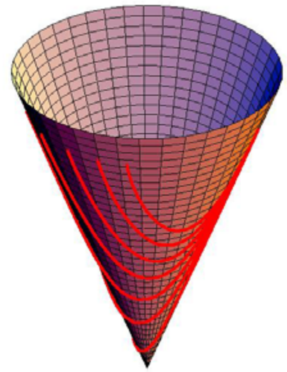

Fig. 2 Solutions on the Cone

The hyperboloid of one-sheet is another interesting surface of revolution parameterized by

$$\mathbf{x}(u,v)=\big(\cosh u\cos v,\cosh u\sin v,\sinh u\big).$$

Figure 3 illustrates several solution curves on the hyperboloid. What is interesting about this surface is the fact that there is a bifurcation of the solutions at C ≈ .687. For values of *C* below the bifurcation point the solution curves do not obtain a minimum value. In fact some of these solution curves go around the hyperboloid and intersect other solution curves. This is illustrated in figure 4. where the red and green curves intersect. The point of intersection can mean one of two things: Either the particle reaches the point of intersection in the same amount of time by following two separate paths or one of the solutions is a local minimum and should be discarded. In the case depicted below the green curve is the local minimum and must be discarded.

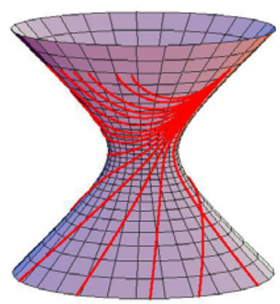

Fig. 3 Solutions on the Hyperboloid.

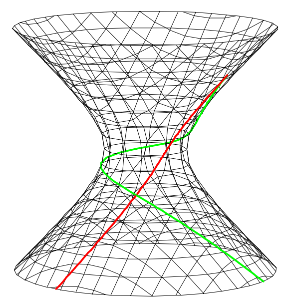

Fig. 4 Wire-frame plot of solution curves.

Inverse Square Fields

Consider a particle confined a plane and falling in an inverse square field. Using polar coordinates we can parameterize the plane by

$$\mathbf{x}(u,v)=(u\cos v,u\sin v).$$

Consequently, $E=1$ and $G=u^2$. Now, letting $V=-u^{-1}$ we can apply the theorem to find the solution curves. Figure 5. illustrates several solution curves for a particle starting at $A=(1,0)$. Again each curve is uniquely determined by *C*. Interestingly, we were able to prove, by taking the limit as $C\to 0$, that no solution curves exist in a 120° sector bisected by the diameter containing the initial point.

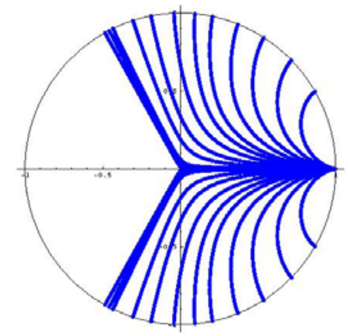

Fig. 5 Planar Solution Curves in an Inverse Square Field

Additionally, if we relax the condition that the particle be confined to a plane, we find that the solution curves are still planar. More specifically, the solution curves lie on a plane uniquely determined by the origin, the starting point, and the final point. Now, if we rotate one of the solution curves in figure 5 we obtain a surface of revolution. The intersection of this surface of revolution and a plane passing through the origin and the point *(1,0)* is another solution curve for a particle starting at *(1,0)*. Figure 6 illustrates several viewpoints of a surface of revolution generated by a solution curve.

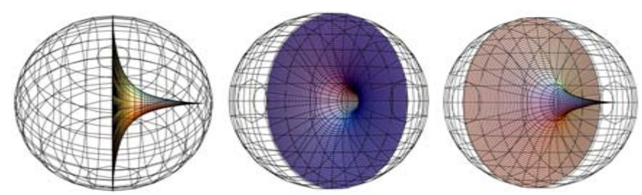

Fig. 6 Several viewpoints of a surface of revolution generated by a solution curve.

Open Questions

We have solved the Brachistochrone problem for a large family of surfaces. But, our current work does not consider the problem for surfaces in fields that require solving a second order equation. We would like to extend our present results to include these types of problems. Our first step towards this goal will be to attempt to find solution curves in dipole fields confined to the plane $z=0$. Then, as we did for the inverse square field, we would like to solve the problem for a particle falling in three dimensions.