Diagonal Approximations on an n-gon and the Cohomology Ring of Closed Compact Orientable Surfaces

A Senior Thesis Submitted to the Department of Mathematics In Partial Fulfillment of the Requirements for the Departmental Honors Baccalaureate

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Abstract

This thesis is a study of diagonal approximations on an n-gon. We will define a linear function Δ on the cellular chains of an n-gon P and show that Δ satisfies the criterion for a diagonal approximation. We will provide a geometric representation of Δ as a tessellation of P. Furthermore, we will show that Δ is a non-coassociative comultiplication. By identifying the appropriate sides of a $4n$-gon P_n, we obtain a closed compact orientable surface T_n of genus n. Think of this identification as a quotient map $T_n^* : P_n \to T_n$ and dualize the cellular chains of T_n to obtain its cellular cochains $C^*(T_n)$. The diagonal Δ induces a multiplication and a differential graded ring structure on $C^*(T_n)$ with trivial differential. Consequently, the cohomology ring of T_n is exactly the ring $C^*(T_n)$.

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Contents

1 Introduction

The diagonal on a set S is the subset $\Delta(S) = \{(x, x) \in S \times S \mid x \in S\}$. This thesis is a study of diagonals on polygons and topological "approximations" thereof. This problem arose from the work of my thesis advisor, Dr. Ronald Umble. His work dealt with providing a diagonal approximation on the associahedra and permutahedra [4],[7], and I began thinking of a way to apply his ideas to find a diagonal approximation of any n-gon. Also, in studying his work, I noticed that all of these approximations involved two disjoint directed paths from a minimal vertex to a maximal vertex. When n is even, both paths have the same number of piecewise components; otherwise, the number of components in these paths differs by one. In this thesis we construct diagonal approximations with arbitrarily chosen distinct minimal and maximal vertices.

We begin the thesis by defining a diagonal approximation Δ on the set of cellular chains of any polygon. We will show that Δ satisfies the criterion for a diagonal approximation. Moreover, we find that Δ is non-coassociative and non-cocommutative. We will also look at the tessellation induced by Δ. This results in some interesting diagrams. We then look at the multiplication induced by Δ on closed compact orientable surfaces, the n-holed tori and 2-sphere. While the original diagonal approximation was non-associative and non-commutative comultiplication, the multiplication induced on closed compact orientable surfaces forms a graded commutative ring with identity.

2 Key Definitions and Background Information

Let X be a polygon. The *geometric diagonal on* X is the function $\Delta_g : X \to X \times X$ given by $\Delta_g(x) = x \times x$. A *cell* of a polygon X is a vertex, edge, or (the single) region of X. Let X and Y be polyhedra. A continuous map $f : X \to Y$ is a *cellular map* if

- vertices of X map to vertices of Y
- edges of X map to edges or vertices of Y
- the region of X maps to the region, an edge, or a vertex of Y.

Definition 1 *Two maps* $f, g : X \to Y$ *are* homotopic *if there is a continuous map* $H : X \times [0,1] \to Y$ *such that* $H(x,0) = f(x)$ *and* $H(x,1) = g(x)$ *for all* $x \in X$.

Definition 2 *Let* X *be a polyhedron. A function* $\Delta : X \to X \times X$ *is a* diagonal approximation *of* Δ_g *if* Δ *is a cellular map homotopic to* Δ_g.

Since every continuous map of polygons is homotopic to a cellular map (see Corollary 16.11 in [2]) and the geometric diagonal Δ_g is continuous, there is a diagonal approximation Δ of Δ_g.

Definition 3 *Let* $\{V_k\}_{k\geq 0}$ *be a collection of vector spaces over a field* F. *The vector space* $V = \oplus_{k\geq 0} V_k$ *is called a* graded vector space*; elements of* V_k *are* homogeneous of degree k.

Definition 4 *A* differential *on a graded vector space* V *is a linear map* $d : V \to V$ *that lowers degree by one. A* differential graded vector space *(DGVS) is a pair* (V, d), *where* d *is a differential on* V.

Definition 5 *Let* X *be a polygon. The vector space* $C_k(X)$ *with basis the* k*-cells of* X *is called the* space of cellular k-chains of X; *the direct sum*

$$C_*(X) = \bigoplus\nolimits_{k\geq 0} C_k(X)$$

is the graded vector space of cellular chains of X.

Definition 6 *Let* X *be a polygon. A* diagonal *on* $C_*(X)$ *is a linear map* $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$ *such that*

1) *If* e *is a cell of* X, *then* $\Delta(e) \subset C_*(e) \otimes C_*(e)$.

2) $\Delta\partial = (\partial \otimes 1 + 1 \otimes \partial)\Delta$.

A diagonal approximation Δ on X induces a diagonal on $C_*(X)$.

The geometric boundary of a k-face $e \subset X$ is a union of $(k-1)$-cells of X. This induces a boundary map $\partial : C_k(X) \to C_{k-1}(X)$ by defining ∂ on each face (the basis) and extending linearly. It is geometrically obvious that $\partial \circ \partial = 0$. Thus the boundary operator ∂ is a differential on $C_*(X)$ and the pair $(C_*(X), \partial)$ is a DGVS.

Definition 7 *Let* (V, d_V) *and* (W, d_W) *be DG vector spaces. A linear map* $f : V \to W$ *is a* chain map *if* $d_W \circ f = f \circ d_V$.

Now consider cellular chains $C_*(P)$ on an n-gon P oriented counterclockwise, with vertices $x_1, x_2, ..., x_m, ..., x_n$ and edges $1, 2, ..., n$, where edge $i = [x_i, x_{i+1}]$ and $x_{n+1} = x_1$. Choose x_1 arbitrarily, and number the vertices cyclically. To define a diagonal $\Delta : C_*(P) \to C_*(P) \otimes C_*(P)$, choose x_1 as the minimal vertex and x_m as the maximal vertex such that x_1 and x_m are disjoint (except in the case of the 1-gon) and define

$$\Delta(x_i) = x_i \otimes x_i$$

$$\Delta(i) = x_i \otimes i + i \otimes x_{i+1}$$

$$\Delta(P) = x_1 \otimes P + P \otimes x_m + \sum_{j=1}^{m-2} \sum_{i=j}^{m-2} j \otimes (i+1) - \sum_{k=m}^{n} \sum_{l=k+1}^{n} l \otimes k. \quad (1)$$

For example, if Q is a 6-gon with maximal vertex x_4, the definition above gives

$$\Delta(Q) = x_1 \otimes Q + Q \otimes x_4 + 1 \otimes 2 + 1 \otimes 3 + 2 \otimes 3 - 6 \otimes 5 - 6 \otimes 4 - 5 \otimes 4.$$

If we choose the maximal vertex to be x_5, Δ yields

$$\Delta(Q) = x_1 \otimes Q + Q \otimes x_5 + 1 \otimes 2 + 1 \otimes 3 + 1 \otimes 4 + 2 \otimes 3 + 2 \otimes 4 + 3 \otimes 4 - 6 \otimes 5.$$

If we choose the maximal vertex to be x_6, Δ yields

$$\Delta(Q) = x_1 \otimes Q + Q \otimes x_5 + 1 \otimes 2 + 1 \otimes 3 + 1 \otimes 4 + 1 \otimes 5 + 2 \otimes 4 + 2 \otimes 5 + 3 \otimes 4 + 3 \otimes 5 + 4 \otimes 5.$$

Because numbering the edges and vertices is arbitrary, Δ with maximal vertices x_2 and x_3 are given by symmetry.

The diagonal Δ can be represented geometry by transforming Δ into a cellular inclusion map. This is done by tessellating the n-gon into two n-gons and a number of 4-gons. This tessellation will be discussed in section 4 of the thesis.

As we shall see, the diagonal defined above is a comultipication Δ on $C_*(P)$. Let y be a cell of the n-gon P. Define a linear function $y^* : C_*(P) \to F$ by

$$y^*(x) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

These functions form a basis for the vector space $C^*(P)$ of cellular cochains of P, which is the linear dual of $C_*(P)$. The diagonal Δ induces an associative multiplication called the cup product. We will prove this strict associativity later in the thesis when we discuss the cohomology ring of a closed compact orientable surface.

Definition 8 *Let X be an cell complex and let y be a cell of X. Then $C^i(X) = \{y^* \mid y \in C_i(X)\}$ is the vector space of* cellular i-cochains of X. *A diagonal approximation Δ_X on X induces a multiplication on $C^*(X)$, called the* cup product, *defined on a basis element $x \in C_*(X)$ by*

$$(a^* \smile b^*)(x) = m(a^* \otimes b^*)\Delta_X(x),$$

where m denotes multiplication in F.

The cup product becomes very interesting when applied to the n-holed torus and will be discussed later in the paper.

3 Examination of Diagonal Approximations on Polygons

In this section, we will prove that each diagonal approximation Δ defined above induces a diagonal on the cellular chains of the polygon. Except in the case of a 1-gon, we will assume that the minimal and maximal vertices are distinct.

Theorem 9 *Each diagonal approximation Δ on a polygon P induces a diagonal on $C_*(P)$.*

Proof. By definition, Δ satisfies the first condition for a diagonal approximation on $C_*(P)$. Let us now consider the second condition. First, we will show the condition holds for a 1-gon and a 2-gon. In the case of the 1-gon P_1 with vertex v and edge 1:

$$\begin{aligned}\Delta(\partial(P_1)) &= \Delta(1) = v \otimes 1 + 1 \otimes v \\ &= (\partial \otimes 1 + 1 \otimes \partial)(v \otimes P_1 + P_1 \otimes v) \\ &= (\partial \otimes 1 + 1 \otimes \partial)\Delta(P_1).\end{aligned}$$

Now we will show the condition holds for the 2-gon P_2 with vertices v_1 and v_2 and edges 1 and 2.

$$\begin{aligned}\Delta(\partial(P)) &= \Delta(1-2) = v_1 \otimes 1 + 1 \otimes v_2 - v_2 \otimes 2 - 2 \otimes v_2 \\ &= (\partial \otimes 1 + 1 \otimes \partial)(v_1 \otimes P_2 + P_2 \otimes v_2) \\ &= (\partial \otimes 1 + 1 \otimes \partial)\Delta(P_2).\end{aligned}$$

The proof for cases with $n \geq 3$ is by induction. The basic step is the Alexander-Whitney diagonal on the triangle [3]. Denote the vertices of a triangle by 0, 1, and 2. Denote the edges by 01, 02, and 12. Denote the region within the triangle by 012, oriented counterclockwise. Without loss of generality, assume that 0 is the minimal vertex and 2 is the maximal vertex.

$$\begin{aligned}\Delta(\partial(012)) &= \Delta(01 + 12 - 02) \\ &= 0 \otimes 01 + 01 \otimes 1 + 1 \otimes 12 + 12 \otimes 2 - 0 \otimes 02 - 02 \otimes 2 \\ &\qquad -0 \otimes 02 - 01 \otimes 2 + 01 \otimes 1 \\ &= 1 \otimes 12 - 0 \otimes 12 + 01 \otimes 2 + 12 \otimes 2 - 02 \otimes 2 \\ &\qquad +0 \otimes 01 + 0 \otimes 12 \\ &= (1-0) \otimes 12 + (01 + 12 - 02) \otimes 2 + 0 \otimes (01 + 12 - 01) \\ &\qquad -01 \otimes (2-1) \\ &= (\partial \otimes 1 + 1 \otimes \partial)(0 \otimes 012 + 01 \otimes 12 + 012 \otimes 2) \\ &= (\partial \otimes 1 + 1 \otimes \partial)(\Delta(012)).\end{aligned}$$

■

Now let us assume that the statement in the theorem is true for all k-gons where $k < n$. Consider two cases. In case 1, assume that x_1 and x_m are not adjacent. Construct the edge $y = [x_1, x_m]$, decomposing P as the union of a j-gon Q and k-gon R, where $j, k < n$. We will first prove two lemmas.

Lemma 10 $\partial(P) = \partial(Q) + \partial(R)$.

Proof.

$$\begin{aligned}
\partial(Q) + \partial(R) &= 1 + 2 + \cdots + (m-1) - y + y - m - (m+1) - \cdots - n \\
&= 1 + 2 + \cdots + (m-1) - m - (m+1) - \cdots - n \\
&= \partial(P)
\end{aligned}$$

■

Lemma 11 $\Delta(P) = \Delta(Q) + \Delta(R)$.

Proof.

$$\begin{aligned}
\Delta(Q) + \Delta(R) &= x_1 \otimes Q + Q \otimes x_m + 1 \otimes 2 + \cdots + (m-2) \otimes (m-1) \\
&+ x_1 \otimes R + R \otimes x_m - n \otimes (n-1) - \cdots - (m+1) \otimes m \\
&= x_1 \otimes (Q+R) + (Q+R) \otimes x_m + 1 \otimes 2 + \cdots \\
&\qquad +(m-2) \otimes (m-1) - n \otimes (n-1) - \cdots - (m+1) \otimes m \\
&= x_1 \otimes P + P \otimes x_m + 1 \otimes 2 + \cdots \\
&\qquad +(m-2) \otimes (m-1) - n \otimes (n-1) - \cdots - (m+1) \otimes m \\
&= \Delta(P).
\end{aligned}$$

■

Now that we have these two lemmas, I am going to prove case 1. By Lemma 10, the induction hypothesis, and Lemma 11 we have

$$\begin{aligned}
\Delta(\partial(P)) &= \Delta(\partial(Q) + \partial(R)) \\
&= \Delta(\partial(Q)) + \Delta(\partial(R)) \\
&= (\partial \otimes 1 + 1 \otimes \partial)\Delta(Q) + (\partial \otimes 1 + 1 \otimes \partial)\Delta(R) \\
&= (\partial \otimes 1 + 1 \otimes \partial)(\Delta(Q) + \Delta(R)) \\
&= (\partial \otimes 1 + 1 \otimes \partial)\Delta(P).
\end{aligned}$$

This completes the proof of case 1.

In case 2, we look at what happens when x_1 and x_m are adjacent to each other ($m = n$). To begin, construct a point X inside the polygon P. Construct two edges, $a = [x_1, X]$ and $b = [X, x_m]$. This splits P into a $(n+1)$-gon S (where the minimal and maximal vertices are not adjacent) and a triangle T. Again, we will first need to prove two lemmas.

Lemma 12 $\partial(P) = \partial(S) + \partial(T)$.

Proof.

$$\begin{aligned}\partial(S) + \partial(T) &= 1 + 2 + \cdots + (n-1) - a - b + a + b - n \\ &= 1 + 2 + \cdots + (n-1) - n \\ &= \partial(P).\end{aligned}$$

■

Lemma 13 $\Delta(P) = \Delta(S) + \Delta(T)$.

Proof.

$$\begin{aligned}\Delta(S) + \Delta(T) &= x_1 \otimes S + S \otimes x_n + 1 \otimes 2 + \cdots \\ &\quad + (n-2) \otimes (n-1) - a \otimes b + x_1 \otimes T + T \otimes x_n + a \otimes b \\ &= x_1 \otimes (S+T) + (S+T) \otimes x_n + 1 \otimes 2 + \cdots + (n-2) \otimes (n-1) \\ &= x_1 \otimes (P) + (P) \otimes x_n + 1 \otimes 2 + \cdots + (n-2) \otimes (n-1) \\ &= \Delta(P).\end{aligned}$$

■

Now that we have Lemmas 12 and 13, let us prove case 2. By Lemma 12, the induction hypothesis, the proof of case 1, and Lemma 13 we have

$$\begin{aligned}\Delta(\partial(P)) &= \Delta(\partial(S) + \partial(T)) \\ &= \Delta(\partial(S)) + \Delta(\partial(T)) \\ &= (\partial \otimes 1 + 1 \otimes \partial)\Delta(S) + (\partial \otimes 1 + 1 \otimes \partial)\Delta(T) \\ &= (\partial \otimes 1 + 1 \otimes \partial)(\Delta(S) + \Delta(T)) \\ &= (\partial \otimes 1 + 1 \otimes \partial)\Delta(P).\end{aligned}$$

This completes the proof of case 2.

Since Δ is a chain map, we can look at whether or not Δ is coassociative. By the general theory of contractable polygons, Δ is coassociative up to homotopy. In the case of the 1-gon and 2-gon, Δ is strictly coassociative. Also, since Δ is a generalization of the Alexander-Whitney diagonal on the 3-gon [3] and the Serre diagonal on the 4-gon [5], we know that, in these cases, Δ is strictly coassociative. However, this is not true for the general diagonal.

Theorem 14 *If P is a 4-gon where the minimal and maximal vertices are adjacent, then Δ is not coassociative.*

Proof. Consider the 4-gon P where x_1 is the minimal vertex and x_4 is the maximal vertex.

$$\begin{aligned}
(\Delta \otimes 1 - 1 \otimes \Delta)\Delta(P) &= (\Delta \otimes 1 - 1 \otimes \Delta)(x_1 \otimes P + P \otimes x_4 + 1 \otimes 2 + 1 \otimes 3 + 2 \otimes 3) \\
&= x_1 \otimes x_1 \otimes P - P \otimes x_4 \otimes x_4 + (x_1 \otimes 1 + 1 \otimes x_2) \otimes 2 \\
&\quad +(x_1 \otimes P + P \otimes x_4 + 1 \otimes 2 + 1 \otimes 3 + 2 \otimes 3) \otimes x_4 \\
&\quad -1 \otimes (x_2 \otimes 2 + 2 \otimes x_3) - 1 \otimes (x_3 \otimes 3 + 3 \otimes x_4) \\
&\quad -2 \otimes (x_3 \otimes 3 + 3 \otimes x_4) \\
&\quad +(x_1 \otimes 1 + 1 \otimes x_2) \otimes 3 + (x_2 \otimes 2 + 2 \otimes x_3) \otimes 3 \\
&\quad -x_1 \otimes (x_1 \otimes P + P \otimes x_4 + 1 \otimes 2 + 1 \otimes 3 + 2 \otimes 3) \\
&= 1 \otimes 2 \otimes x_3 + 1 \otimes x_2 \otimes 3 + x_2 \otimes 2 \otimes 3 - x_1 \otimes 2 \otimes 3 \\
&\quad -1 \otimes 2 \otimes x_3 - 1 \otimes x_3 \otimes 3 \\
&\neq 0.
\end{aligned}$$

■

Now we will consider the case where we have an n-gon P with $n > 4$.

Theorem 15 *For $n > 4$, Δ is not strictly coassociative.*

Proof. The basic step is the general 5-gon.

First, consider the case when the minimal and maximal vertices are not adjacent.

$$\begin{aligned}
(\Delta \otimes 1 - 1 \otimes \Delta)(\Delta(P)) &= (\Delta \otimes 1 - 1 \otimes \Delta)(x_1 \otimes P + P \otimes x_3 + 1 \otimes 2 \\
&\quad -5 \otimes 4 - 5 \otimes 3 - 4 \otimes 3) \\
&= x_1 \otimes x_1 \otimes P + (x_1 \otimes P + P \otimes x_3 + 1 \otimes 2 \\
&\quad -5 \otimes 4 - 5 \otimes 3 - 4 \otimes 3) \otimes x_3 + (x_1 \otimes 1 + 1 \otimes x_2) \otimes 2 \\
&\quad -(x_1 \otimes 5 + 5 \otimes x_5) \otimes 4 - (x_1 \otimes 5 + 5 \otimes x_5) \otimes 3 \\
&\quad -(x_1 \otimes 5 + 5 \otimes x_5) \otimes 4 - (x_5 \otimes 4 + 4 \otimes x_4) \otimes 3 \\
&\quad -x_1 \otimes (x_1 \otimes P + P \otimes x_3 + 1 \otimes 2 - 5 \otimes 4 - 5 \otimes 3 - 4 \otimes 3) \\
&\quad -P \otimes x_3 \otimes x_3 - 1 \otimes (x_2 \otimes 2 + 2 \otimes x_3) \\
&\quad +5 \otimes (x_5 \otimes 4 + 4 \otimes x_4) + 5 \otimes (x_4 \otimes 3 + 3 \otimes x_3) \\
&\quad +4 \otimes (x_4 \otimes 3 + 3 \otimes x_3) \\
&= x_1 \otimes 4 \otimes 3 + 5 \otimes 4 \otimes x_4 + 5 \otimes x_4 \otimes 3 - 5 \otimes 4 \otimes x_3 \\
&\quad -5 \otimes x_4 \otimes 3 - x_5 \otimes 4 \otimes 3 \\
&\neq 0.
\end{aligned}$$

Now, consider the case where minimal and maximal vertices are adjacent.

$$\begin{aligned}
(\Delta \otimes 1 - 1 \otimes \Delta)(\Delta(P)) &= (\Delta \otimes 1 - 1 \otimes \Delta)(x_1 \otimes P + P \otimes x_5 + 1 \otimes 2 \\
&\quad +1 \otimes 3 + 1 \otimes 4 + 2 \otimes 3 + 2 \otimes 4 + 3 \otimes 4) \\
&= x_1 \otimes x_1 \otimes P + (x_1 \otimes P + P \otimes x_5 + 1 \otimes 2 \\
&\quad +1 \otimes 3 + 1 \otimes 4 + 2 \otimes 3 + 2 \otimes 4 + 3 \otimes 4) \otimes x_5 \\
&\quad +(x_1 \otimes 1 + 1 \otimes x_2) \otimes 2 + (x_1 \otimes 1 + 1 \otimes x_2) \otimes 3 \\
&\quad +(x_1 \otimes 1 + 1 \otimes x_2) \otimes 4 + (x_2 \otimes 2 + 2 \otimes x_3) \otimes 3 \\
&\quad +(x_2 \otimes 2 + 2 \otimes x_3) \otimes 4 + (x_3 \otimes 3 + 3 \otimes x_4) \otimes 4 \\
&\quad -x_1 \otimes x_1 \otimes P - (x_1 \otimes P + P \otimes x_5 + 1 \otimes 2 + 1 \otimes 3 \\
&\quad +1 \otimes 4 + 2 \otimes 3 + 2 \otimes 4 + 3 \otimes 4) \otimes x_5 \\
&\quad -1 \otimes (x_2 \otimes 2 + 2 \otimes x_3) - 1 \otimes (x_3 \otimes 3 + 3 \otimes x_4)
\end{aligned} \tag{2}$$

$$
\begin{aligned}
&-1 \otimes (x_4 \otimes 4 + 4 \otimes x_5) - 2 \otimes (x_3 \otimes 3 + 3 \otimes x_4) \\
&-2 \otimes (x_4 \otimes 4 + 4 \otimes x_5) - 3 \otimes (x_4 \otimes 4 + 4 \otimes x_5) \\
= \quad & 1 \otimes 2 \otimes x_5 + 1 \otimes 3 \otimes x_5 + 2 \otimes 3 \otimes x_5 + 1 \otimes x_2 \otimes 3 \\
&+1 \otimes x_2 \otimes 4 + x_2 \otimes 2 \otimes 3 + x_2 \otimes 2 \otimes 4 + 2 \otimes x_3 \otimes 4 \\
&+x_3 \otimes 3 \otimes 4 - x_1 \otimes 2 \otimes 3 - x_1 \otimes 2 \otimes 4 - x_1 \otimes 3 \otimes 4 \\
&-1 \otimes 2 \otimes x_3 - 1 \otimes x_3 \otimes 3 - 1 \otimes x_4 \otimes 4 - 2 \otimes 3 \otimes x_4 \\
&-2 \otimes x_4 \otimes 4 - 1 \otimes x_4 \otimes 4 \\
\neq \quad & 0.
\end{aligned}
$$

Also, because of the induction proof, I need to consider separately the case of the 6-gon when the maximal vertex is x_4.

$$
\begin{aligned}
(\Delta \otimes 1 - 1 \otimes \Delta)(\Delta(P)) \quad = \quad & (\Delta \otimes 1 - 1 \otimes \Delta)(x_1 \otimes P + P \otimes x_4 \\
&+1 \otimes 2 + 1 \otimes 3 + 2 \otimes 3 - 6 \otimes 5 - 6 \otimes 4 - 5 \otimes 4) \\
= \quad & x_1 \otimes x_1 \otimes P + (x_1 \otimes P + P \otimes x_4 \\
&+1 \otimes 2 + 1 \otimes 3 + 2 \otimes 3 - 6 \otimes 5 - 6 \otimes 4 - 5 \otimes 4) \\
&+(x_1 \otimes 1 + 1 \otimes x_2) \otimes 2 + (x_1 \otimes 1 + 1 \otimes x_2) \otimes 3 \\
&+(x_2 \otimes 2 + 2 \otimes x_3) \otimes 3 - (x_1 \otimes 6 + 6 \otimes x_6) \otimes 5 \\
&-(x_1 \otimes 6 + 6 \otimes x_6) \otimes 4 - (x_6 \otimes 5 + 5 \otimes x_5) \otimes 4 \\
&-x_1 \otimes (x_1 \otimes P + P \otimes x_4 + 1 \otimes 2 + 1 \otimes 3 + 2 \otimes 3 \\
&-6 \otimes 5 - 6 \otimes 4 - 5 \otimes 4) - P \otimes x_4 \otimes x_4 \\
&-1 \otimes (x_2 \otimes 2 + 2 \otimes x_3) - 1 \otimes (x_3 \otimes 3 + 3 \otimes x_4) \\
&-2 \otimes (x_3 \otimes 3 + 3 \otimes x_4) + 6 \otimes (x_6 \otimes 5 + 5 \otimes x_5) \\
&+6 \otimes (x_5 \otimes 4 + 4 \otimes x_4) + 5 \otimes (x_5 \otimes 4 + 4 \otimes x_4)
\end{aligned}
$$

$$
\begin{aligned}
&= 1\otimes 2\otimes x_4 - 6\otimes 5\otimes x_4 + 1\otimes x_2\otimes 3 \\
&\quad +x_2\otimes 2\otimes 3 - 6\otimes x_6\otimes 4 - x_6\otimes 5\otimes 4 \\
&\quad -x_1\otimes 2\otimes 3 - x_1\otimes 5\otimes 4 - 1\otimes 2\otimes x_3 \\
&\quad -1\otimes x_3\otimes 3 + 6\otimes 5\otimes x_6 + 6\otimes x_5\otimes 4 \\
&\neq 0.
\end{aligned} \tag{3}
$$

Next, assume that the diagonal is not strictly coassociative on all k-gons where $k < n$ and show that the n-gon P is not coassociative. First, consider the case where the minimal and maximal vertices are not adjacent. We know that

$$\Delta(P) = \Delta(Q) + \Delta(R),$$

where both Q and R have less than n veritices, by Lemma 11. By the induction hypothesis, we know that at least one of $\Delta(Q)$ or $\Delta(R)$ is not strictly coassociative. Also, $\Delta(Q)$ and $\Delta(R)$ share no common factors because of the way that Q and R were constructed. Therefore, $\Delta(P)$ is not strictly coassociative.

Next, consider the case where the minimal and maximal vertices are adjacent. Construct the n-gon by subdividing the edge $n-1$ of an (n-1)-gon into two edges $[x_{n-1}, a]$ and $[a, x_n]$. Splitting the $(n-1)$-gon in this way does not affect the non-strictly coassociative nature of Δ since adding the edge can only add terms when we consider $(\Delta \otimes 1 - 1 \otimes \Delta)(\Delta(P))$. Therefore, Δ is not strictly coassociative on the n-gon. ■

4 Geometric Representation of Δ

In addition to being represented algebraically, Δ can also be represented graphically by partitioning the polygon. To partition the original n-gon P with minimal vertex x_1 and maximal vertex x_m, start with the terms $x_1 \otimes P$ and $P \otimes x_m$. These produce two smaller n-gons: one denoted $x_1 \otimes P$ and the other $P \otimes x_m$, both of which are oriented the same way as the larger n-gon. The the two n-gons meet at the maximal vertex of $x_1 \otimes P$ and the minimal vertex of $P \otimes x_m$. The rest of the terms of Δ can be thought of as the Cartesian product of two line segments as indicated in Figure 1. Below are some examples of Δ on different polygons.

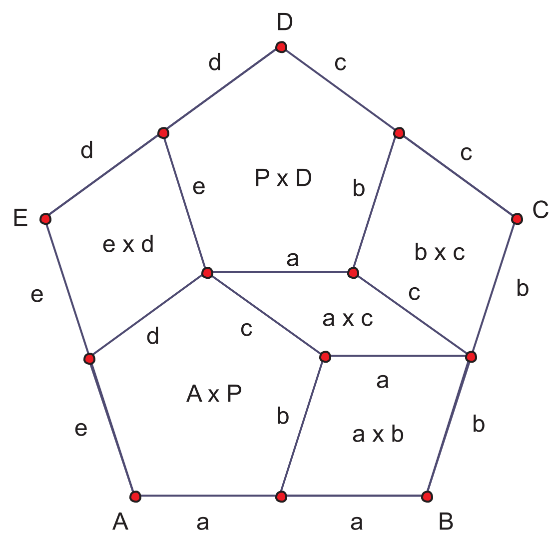


Figure 1: The 5-gon

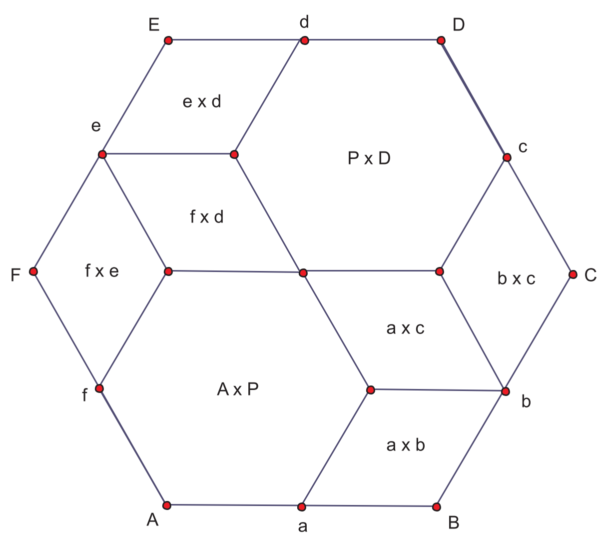


Figure 2: The 6-gon

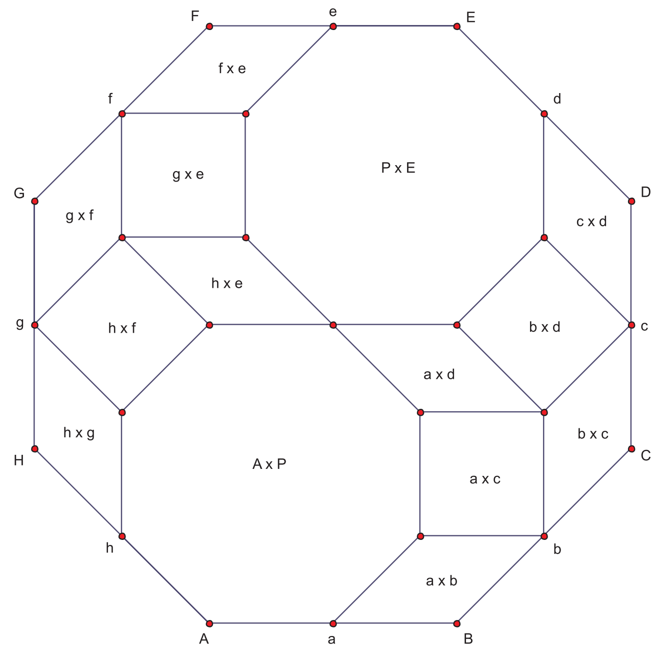
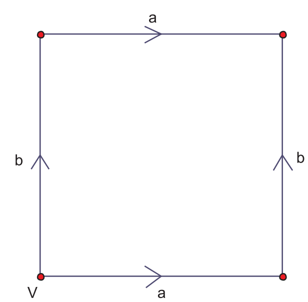


Figure 3: The 8-gon

5 Application: The Cohomology Ring on the Set of Closed Compact Orientable Surfaces

Homeomorphism classes of closed compact orientable surfaces are represented by the 2-sphere and n-holed tori [1]. The number n is called the *genus*. We will consider the cohomology ring of the 2-sphere and n-holed tori independently.

First, consider the case of an n-holed torus. The n-holed torus can be constructed inductively in the following way: To construct the 1-holed torus, identify opposite sides of the square as indicated below.



Note that the identification process identifies all four vertices of the square as a single vertex V on the torus.

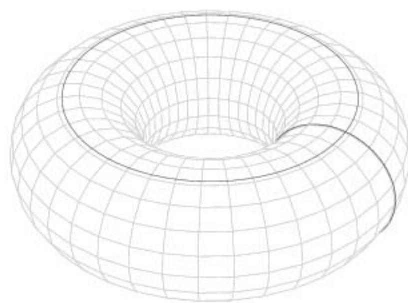
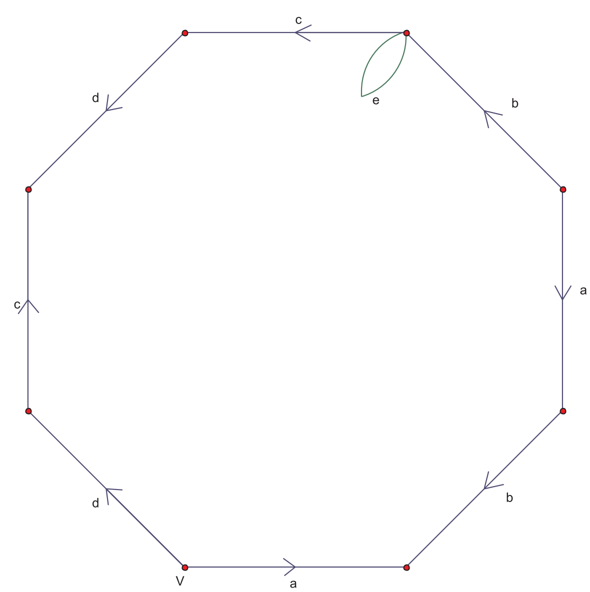


Figure 4: The Torus (from [6])

Now assume the n-holed torus has been constructed by identifying edges of a $4n$-gon P_n. Choose any vertex V of P_n and remove the interior of a 2-dimensional disk tangent to V as indicated in Figure 4 below.



Cut at the vertex V to form a $(4n+1)$-gon with new edge e bounding the disk just removed.

Let P_1 be a square whose opposite sides are identified to form a 1-holed torus. As above, remove the interior of a disk tangent to a vertex V' of P_1. Cut at the vertex V' to form a 5-gon with new edge e' bounding the disk just removed. Now glue the $(4n+1)$-gon and the 5-gon together along edges e and e' to form a $4(n+1)$-gon. The $(n+1)$-holed torus is obtained by performing the identifications indicated in Figure 5 below.

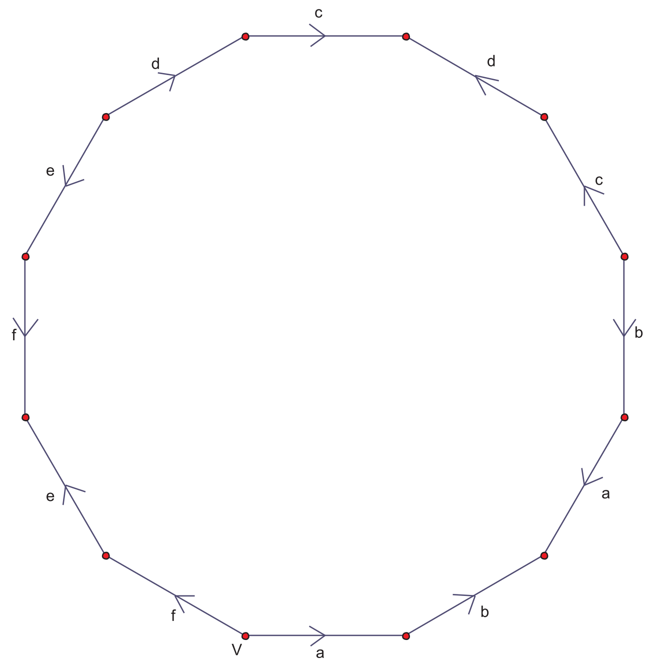


Figure 5: A 12-gon representing a three-holed torus.

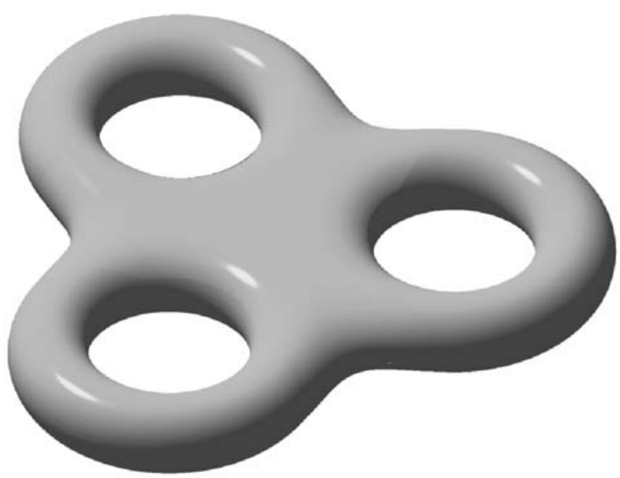


Figure 6: The Three-Holed Torus (from [6])

Definition 16 *Let V be a vector space over a field F. The* dual vector space $V^* = \{$*linear maps* $f : V \to F\}$. *If V is finite dimensional with basis $\{v_1, \cdots, v_n\}$, define $v^* : V \to F$ by*

$$v_i^*(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{v_1^*, \cdots, v_n^*\}$ is a basis for V^*.

A diagonal Δ on $C_*(P_n)$ induces a diagonal on cellular chains of the n-holed torus $C_*(T_n)$. For example, consider the one-holed torus T_1 with vertex V and edges a and b.

$$\Delta(T_1) = V \otimes T_1 + T_1 \otimes V + a \otimes b - b \otimes a.$$

This is different from the Serre diagonal on the square, which has four vertices and four edges. In general, our cellular decomposition of an n-holed torus has one vertex V and edges a_1, b_1, a_2, b_2,..., a_n, b_n. The diagonal on $C_*(T_n)$ induced by the diagonal in line (1) is

$$\begin{aligned} \Delta(T) &= V \otimes T + T \otimes V + a_1 \otimes b_1 - b_1 \otimes a_1 + a_2 \otimes b_2 - b_2 \otimes a_2 + \cdots \\ &\quad + a_n \otimes b_n - b_n \otimes a_n. \end{aligned}$$

This diagonal on T_n induces a multiplication on the dual vector space $C^*(T_n)$, called the cellular cochains of T_n. A basis for $C^*(T_n)$ is $B^* = \{v^*, T_n^*, a_1^*, b_1*, \cdots, a_n^*, b_n^*\}$.

Theorem 17 *$C^*(T_n)$ is a graded commutative ring with identity.*

Proof. First, note that $C^*(T_n)$ is a vector space and hence an additive abelian group. Let B^* be the basis for $C^*(T_n)$ given above. Consider V^*, where V is the vertex of the torus. We check that V^* is an identity element by checking all cases. So we have:

$$\begin{aligned} (V^* \smile V^*)(V) &= m(V^* \otimes V^*)(V \otimes V) \\ &= m(1 \otimes 1) \\ &= 1 \\ &= V^*(V). \end{aligned}$$

Let e be any edge on the torus.

$$\begin{aligned} (V^* \smile e^*)(e) &= m(V^* \otimes e^*)(V \otimes e + e \otimes V)) \\ &= m(1 \otimes 1) \\ &= 1 \\ &= e^*(e). \end{aligned}$$

Finally,

$$\begin{aligned}(V^* \smile T_n^*)(T_n) &= V^* \otimes T_n^*(V \otimes T_n) \\ &= m(1 \otimes 1) \\ &= 1 \\ &= T_n^*(T_n).\end{aligned}$$

Also, because B^* consists only of functions of degree zero, one, or two, functions of degree three or more are the zero function (for degree reasons). Since V^* is the identity element, we need only to check closure for $e_i^* \otimes e_j^* \in B^*$.

$$\begin{aligned}(e_i^* \smile e_j^*)(T_n) &= m(e_i^* \otimes e_j^*(e_i \otimes e_j)) \\ &= m(1 \otimes 1) \\ &= 1 \\ &= V^*(V).\end{aligned}$$

I have included a multiplication table showing the product of any two elements of B^* for an n-holed torus.

$\smile$	v^*	a_1^*	b_1^*	a_2^*	b_2^*	$\cdots$	a_n^*	b_n^*	T_n^*
v^*	v^*	a_1^*	b_1^*	a_2^*	b_2^*	$\cdots$	a_n^*	b_n^*	T_n^*
a_1^*	a_1^*	0	T_n^*	0	0		0	0	0
b_1^*	b_1^*	$-T_n^*$	0	0	0		0	0	0
a_2^*	a_2^*	0	0	0	T_n^*		0	0	0
b_2^*	b_2^*	0	0	$-T_n^*$	0	$\cdots$	0	0	0
$\vdots$	$\vdots$				$\vdots$		$\vdots$		$\vdots$
a_n^*	a_n^*	0	0	0	0	$\cdots$	0	T_n^*	0
b_n^*	b_n^*	0	0	0	0		$-T_n^*$	0	0
T_n^*	T_n^*	0	0	0	0	$\cdots$	0	0	0

For associativity, since V^* is the identity and all functions of degree three or more are the zero function, a nonzero product $a^* \smile (b^* \smile c^*)$ must have degree less than or equal to two. Then, at least one of $a^*, b^*,$, or c^* must be of degree zero. Since the only element of B^* of degree zero is the identity element, $C^*(T_n)$ is multiplicatively associative. Finally, it is straightforward to check that the cup product distributes over addition. Therefore, $C^*(T_n)$ is a graded commutative ring with identity. ∎

The cohomology ring for the 2-sphere S_2^* is formed by mapping the vertex and edge of the 1-gon to a vertex. Define $C^*(S_2)$ as above, except in this case $B^* = \{v^*, S_2^*\}$. The proof that B^* forms a graded commutative ring with identity is similar to the proof given above.

6 Conclusion

Through induction, we were able to show that Δ, defined in (1), is a diagonal on $C^*(P)$. Furthermore, we were able to show that, with the exception of the 1-gon, 2-gon, A-W diagonal and the Serre diagonal, Δ was neither coassociative or cocommutative. The geometric representation of Δ as a tessellation of P provided some very interesting pictures as well. Furthermore, the multiplication $\smile$ induces on $C^*(T_n)$ is a graded commutative ring with identity.

In conclusion, there are many open questions related to diagonal approximations. The diagonal approximation discussed in this thesis is not the only one. Also, we have the cohomology ring of the closed compact orientable surfaces. It would be interesting to find the cohomology ring of the non-orientable surfaces in a similar way. For the tessellations we defined in this thesis, we were unable to find a general formula for the number of smaller polygons that tessellates the larger n-gon. Such a formula would be useful for tessellating larger polygons. Finally, we know that Δ is coassociative up to homotopy, but we do not know what that homotopy is. It would be interesting to find the homotopy for all n if it is possible.

References

[1] E. Block, *A First Course in Geometric Topology and Differential Geometry*, Birkhäuser Press, Boston, 1997.

[2] B. Gray, *Homotopy Theory: An Introduction to Algebraic Topology,* Academic Press, New York, 1975, pp. 133.

[3] S. Mac Lane, "Homology," Springer-Verlag, Berlin/New York, 1967.

[4] S. Saneblidze and R. Umble, Diagonals on the Permutahedra, Multiplihedra and Associahedra, *J. Homology, Homotopy and Appl.*, **6** (1) (2004), 363-411.

[5] J.P. Serre, Homoligie Singuliere des Espaces Fibres, Apllications, Ann. Math., **54** (1951), 439-505.

[6] *Torus.* "Wikipedia, the Free Encyclopedia," April 29, 2008, http://en.wikipedia.org/wiki/Torus.

[7] S. Weaver, Computing the Sameblidze-Umble Diagonal on Permutahedra, Senior thesis, Millersville University, May 2005.