

Millersville University

Distance Minimizing Paths on Certain Simple Flat Surfaces with Singularities

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Abstract

We analyze a conical cup with lid and a circular tin can to find the shortest path connecting two points anywhere on the surface. We show how to transform the problem into an equivalent problem in plane geometry and prove that such paths can be found in most cases.

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1 Introduction

In the spring of 2001, I participated in a research seminar directed by Dr. Ron Umble in which we considered the following problem posed to our group by Dr. Frank Morgan of Williams College: *Given a circular tin can with two points placed anywhere on the surface of the can, find the distance minimizing path constrained to the surface of the can connecting these two points.* The original research group, which included Heather Heston, Ellen Panofsky, Robert Painter and myself, proved that there exist distance minimizing paths in the following cases:

1. Two points on the side of the can,
2. A point on the lid and a point on the side and
3. A point on the lid and a point on the base.

In this thesis we offer simpler proofs of some of the same results and obtain more general results near the rim. We generalize our earlier work to include conical cups with lid (or “cup” for short). We observe that many of our results on the tin can be recovered from similar results on the cup by taking the limit as the cone height goes to infinity.

As one might expect, the problem of finding distance minimizing paths constrained to the surface of a cup is more difficult than on the tin can. Like the tin can, a cup is a closed surface with singularities along the rim where the lid joins the side of the cup. However, the singularity at the cone point is quite different from those on the rim. Indeed, we show that distance minimizing paths connecting two points distinct from the cone point always miss the cone point. Furthermore, geodesics on the cup (not necessarily distance minimizing) have interesting properties not possessed by their helical analogues on the tin can. For example, under the right conditions, geodesics on the side of the cup have self-intersections; this never happens on the side of the tin can. We show that the number of self-intersections is finite and is given by an explicit function of the cone angle.

The stage on which much of our work in this thesis plays out is a flat model of the surface in the Euclidean plane \mathbb{R}^2 . Each surface is mapped onto a region of the plane in which straight lines correspond to geodesics on the surface. This transformation greatly simplifies the analysis.

We have used Wolfram Research's (www.wolfram.com) Mathematica extensively in all stages of this research. Also note that all of the graphics in this thesis were generated by Mathematica with some additional refinements using XFig (www.xfig.org).

2 Simple Flat Surfaces

In this section we give precise definitions of the surfaces we are studying. We think of the tin can or cup as surfaces of revolution about the z -axis in \mathbb{R}^3 with cylindrical coordinates. In most cases we consider a can or cup with unit radius. Results obtained with unit radius extend easily to results with arbitrary radius by scaling.

Definition 1. A simple flat surface in \mathbb{R}^3 is a piecewise smooth surface S , each piece of which is obtained from a region of the plane \mathbb{R}^2 by bending while preserving distance between points. A set of plane regions that produce S when bent and glued appropriately along their boundaries is called a flat model for S .

A cup and tin can are examples of simple flat surfaces.

Definition 2. A conical cup with lid of height h is the union of the following sets:

1. The conical side: $\{(r, \theta, z) \mid r = \frac{z}{h}, 0 \leq \theta < 2\pi \text{ and } 0 < z < h\}$
2. The circular lid: $\{(r, \theta, h) \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta < 2\pi\}$
3. The cone point: $\{(0, 0, 0)\}$.

We use the terms “cup” and “conical cup with lid” synonymously.

Definition 3. A circular can of height h is the union of the following sets:

1. The tubular side: $\{(1, \theta, z) \mid 0 \leq \theta < 2\pi \text{ and } 0 < z < h\}$
2. The circular lid: $\{(r, \theta, h) \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta < 2\pi\}$
3. The circular base: $\{(r, \theta, 0) \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta < 2\pi\}$

The shape of a conical cup is determined by its “cone angle.”

Definition 4. The cone angle of a conical cup is the angle between its axis of revolution and a radial line in the surface of the cone emanating from the cone point.

When the side of a conical cup is cut along a radial line, it can be unrolled and flattened into a sector of a circle. The angle of this sector is determined by the cone angle of the cup.

Definition 5. The plane cone angle of a conical cup is the angle of the sector obtained when the conical surface is cut, unrolled and flattened.

We typically denote the cone angle by α and the plane cone angle by θ . A picture of a cup and a corresponding flat model appear in Figure 1.

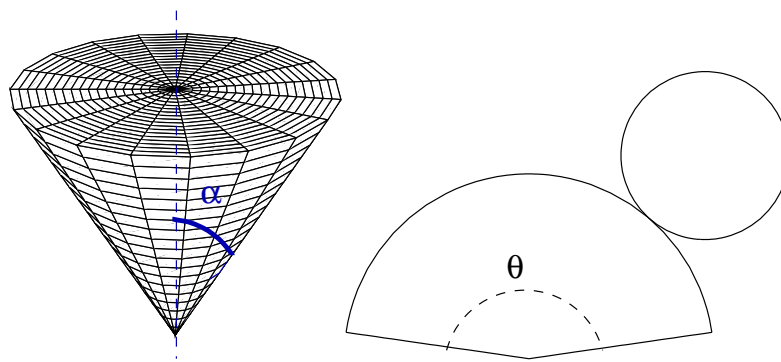


Figure 1: A conical cup and a corresponding flat model

3 Singularities along the Rim

An interesting family of curves, called roulettes, arise from our analysis of flat models. Let C_1 and C_2 be smooth plane curves tangent to each other at some point A . Let $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$ parametrize C_2 by arc length s with $\gamma(0) = A$. Let $\tau_s: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation defined by rolling C_1 along C_2 without slipping until $\gamma(s)$ is the point of tangency.

Definition 6. Let $B \in \mathbb{R}^2$ and let C_1 and C_2 be smooth plane curves tangent to each other at some point. The roulette generated by B is the curve $\tau_s(B)$, $s \in \mathbb{R}$.

When a circle rolls along a line, for example, B generates a special roulette called a cycloid. A picture of a cycloid appears in Figure 2.

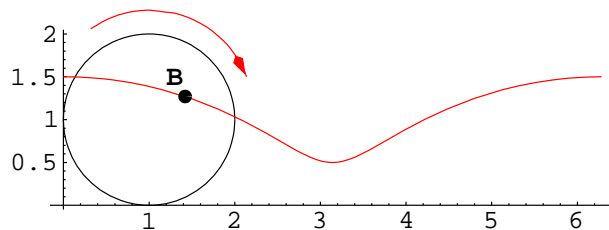


Figure 2: Cycloid being generated by circle rolling along the x -axis

3.1 Roulettes Along Lines

A cycloid arises when we unroll the tin can into a corresponding flat model. The cycloid is generated by a point on the lid of the tin can. This cycloid has the interesting property that the line normal to the cycloid at any given point where the normal line is defined passes through the point of tangency between the lid and the side of the can. In fact, this property holds for roulettes formed by rolling any smooth curve along a straight line.

Theorem 1. *Suppose that $\rho(t)$ is a roulette formed by rolling a curve C along a straight line l . At any point B along the path $\rho(t)$, a line normal to $\rho(t)$ through B will pass through the corresponding point of tangency of C with the line l .*

Proof. Suppose that C is a plane curve parameterized by $\beta(t) = \langle p(t), q(t) \rangle$, not necessarily an arc length parametrization, and suppose that $B_0(x_1, y_1)$ is a point in the plane. Suppose that C is rolled along the x -axis so that $\beta(0)$ corresponds to the x -axis at the origin. For any $t_0 > 0 \in \mathbb{R}$, after C is rolled so that $\beta(t_0)$ is tangent to the x -axis, the point of tangency is determined by the arc-length of β from $t = 0$ to $t = t_0$:

$$\left(\int_0^{t_0} \sqrt{p'(t)^2 + q'(t)^2} dt, 0 \right).$$

The transformation of rolling C along the x -axis maps points on C to points on the x -axis. At every point P' on C , B_0 and P' determine a vector $\overrightarrow{B_0P'}$. When C is rolled along the x -axis, $\overrightarrow{B_0P'}$ is transformed to \overrightarrow{BP} . To find the path of B through all transformations, we look at the components of \overrightarrow{BP} parallel to the axes. Since C is rolling along the x -axis, we know that at the point of tangency, the tangent line to C is coincident with the x -axis. This means that to find the coordinates of $\rho(t)$ we need to find the components of the vector $\gamma(t) = \overrightarrow{B_0P'}$ along $\beta'(t)$ and $\beta'(t)_\perp$.

$$\text{comp}_{\beta'(t)} \gamma(t) = \frac{\gamma(t) \cdot \beta'(t)}{\|\beta'(t)\|}$$

$$\begin{aligned}
&= \frac{\langle x_1 - p(t), y_1 - q(t) \rangle \cdot \langle p'(t), q'(t) \rangle}{\|\beta'(t)\|} \\
&= \frac{(x_1 - p(t))p'(t) + (y_1 - q(t))q'(t)}{\sqrt{p'(t)^2 + q'(t)^2}}
\end{aligned}$$

This result is the adjustment of the x -coordinate to the right or left of the tangent point P' . Analogously we find the component of $\gamma(t)$ along $\beta'(t)_\perp$ to find the y coordinate of $\rho(t)$.

$$\begin{aligned}
\text{comp}_{\beta'(t)_\perp} \gamma(t) &= \frac{\gamma(t) \cdot \beta'(t)_\perp}{\|\beta'(t)_\perp\|} \\
&= \frac{\langle x_1 - p(t), y_1 - q(t) \rangle \cdot \langle -q'(t), p'(t) \rangle}{\|\beta'(t)_\perp\|} \\
&= \frac{-(x_1 - p(t))q'(t) + (y_1 - q(t))p'(t)}{\sqrt{p'(t)^2 + q'(t)^2}}
\end{aligned}$$

When we combine both the x and y coordinates we get the final parametrization of $\rho(t)$:

$$\begin{aligned}
\rho(t) = \left\langle \int_0^t \sqrt{p'(u)^2 + q'(u)^2} du + \frac{(x_1 - p(t))p'(t) + (y_1 - q(t))q'(t)}{\sqrt{p'(t)^2 + q'(t)^2}}, \right. \\
\left. \frac{-(x_1 - p(t))q'(t) + (y_1 - q(t))p'(t)}{\sqrt{p'(t)^2 + q'(t)^2}} \right\rangle.
\end{aligned}$$

With the parametrization of the roulette ρ we can easily find a formula for the slope of the normal line through any point on ρ and we can also find the slope connecting this point on ρ and the corresponding tangent point. Let ρ_x and ρ_y be the first and second components of ρ respectively, and define P_x and P_y analogously for point P .

$$-\frac{\rho'_x(t)}{\rho'_y(t)} = \frac{(y_1 - q(t))p'(t) + (-x_1 + p(t))q'(t)}{(x_1 - p(t))p'(t) + (y_1 - q(t))q'(t)} \quad (1)$$

$$\frac{\rho_y - P_y}{\rho_x - P_x} = \frac{\frac{(y_1 - q(t))p'(t) + (-x_1 + p(t))q'(t)}{D(t)}}{\int_0^t D(u) du + \frac{(x_1 - p(t))p'(t) + (y_1 - q(t))q'(t)}{D(t)} - \int_0^t D(u) du}$$

where $D(u) = \sqrt{p'(u)^2 + q'(u)^2}$.

$$\frac{\rho_y - P_y}{\rho_x - P_x} = \frac{(y_1 - q(t))p'(t) + (-x_1 + p(t))q'(t)}{(x_1 - p(t))p'(t) + (y_1 - q(t))q'(t)} \quad (2)$$

From equations (1) and (2), we see that:

$$-\frac{\rho'_x(t)}{\rho'_y(t)} = \frac{\rho_y - P_y}{\rho_x - P_x}$$

This means that the slope of the normal line is identically the same as the slope of the line formed by connecting the tangent point and the corresponding point on the roulette. Since these slopes are equal and both lines share a common point on the roulette, they must be the same line. This completes the proof that the normal line to the roulette passes through the point of tangency. \square

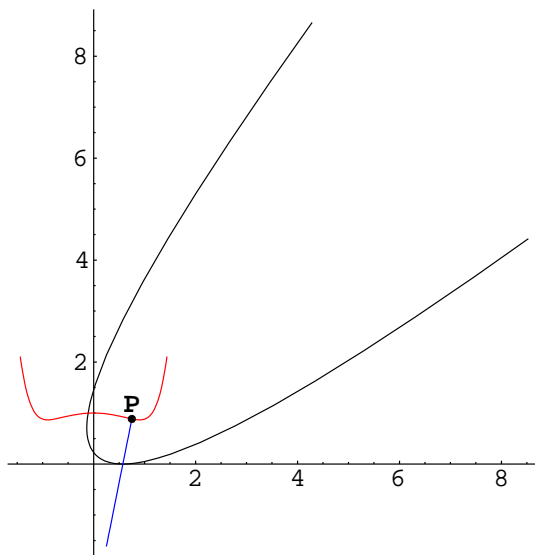


Figure 3: Roulette with normal line generated by the point $(0, 1)$ when rolling the parabola $y = x^2$ along the x -axis

An example of the normal line passing through the tangent point is shown in Figure 3.

3.2 Roulettes Along Circles

Just as a cycloid arises from the flat model of the tin can, an epicycloid is generated by a point on the lid of the cup as the lid rolls around the circular edge of the unrolled side. An epicycloid is a roulette generated by a point in the plane of one circle rolling around another circle.

In this thesis, we will be using the parametrization of the epicycloid shown below. We will be using a circle of radius s centered at $(0, -s)$ in the xy plane. Such a picture can be seen in Figure 4.

$$\beta(t) = \left\langle (1+s) \sin \frac{t}{s} - b \sin \left(\frac{t}{s} + t \right), -s + (1+s) \cos \frac{t}{s} - b \cos \left(\frac{t}{s} + t \right) \right\rangle \quad (3)$$

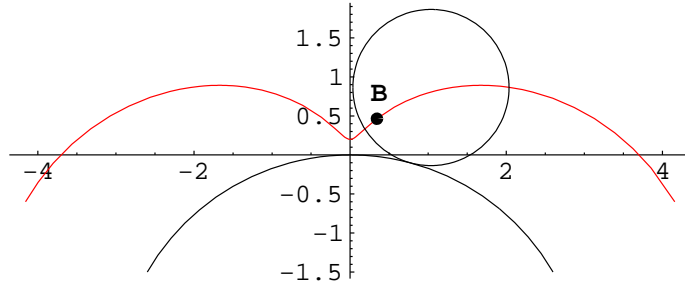


Figure 4: Epicycloid Parametrization

In Theorem 1, we showed that the normal line to a roulette formed by rolling a general smooth curve along a line passes through the corresponding point of tangency. Since the epicycloid is formed by rolling a circle around another circle, this theorem does not apply in this case. We can state an analogous result in the specific case of the epicycloid.

Theorem 2. *Let $\beta(t)$ be an epicycloid formed by rolling a circle around another circle. For every point B along $\beta(t)$, wherever the normal line is defined, the line normal to $\beta(t)$ through B will pass through the point of tangency corresponding to B .*

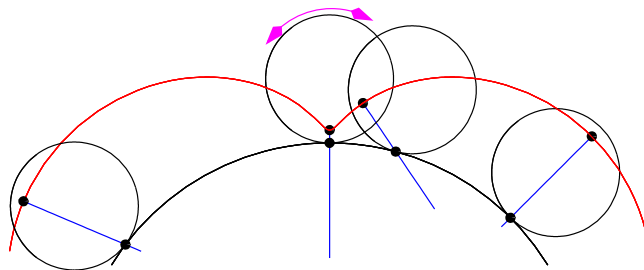


Figure 5: Epicycloid normal lines passing through the point of tangency

Proof. The proof is straightforward. Let $\beta(t)$ parametrize an epicycloid about the circle of radius s centered at $(0, -s)$. We compute the slope of the normal line by using the derivative of $\beta(t)$ and then compare this slope to the slope determined by the two points B and the point of tangency corresponding to B .

$$\beta(t) = \left\langle (1+s) \sin \frac{t}{s} - b \sin \left(\frac{t}{s} + t \right), -s + (1+s) \cos \frac{t}{s} - b \cos \left(\frac{t}{s} + t \right) \right\rangle$$

If we parametrize the circular arc of the side as $\gamma(t) = \langle s \sin \frac{t}{s}, -s + s \cos \frac{t}{s} \rangle$, $\gamma(t)$ is the point of tangency when the epicycloid is at $\beta(t)$. We can then compute the slope from any arbitrary point $\beta(t)$ to $\gamma(t)$ by:

$$\frac{-s + (1+s) \cos \frac{t}{s} - b \cos \left(\frac{t}{s} + t \right) - [-s + s \cos \frac{t}{s}]}{(1+s) \sin \frac{t}{s} - b \sin \left(\frac{t}{s} + t \right) - s \sin \frac{t}{s}}$$

or simplifying yields

$$\frac{\cos \frac{t}{s} - b \cos \left(\frac{t}{s} + t \right)}{\sin \frac{t}{s} - b \sin \left(\frac{t}{s} + t \right)}. \quad (4)$$

The slope of the normal line can be calculated by taking the negative reciprocal of the slope of the line tangent to the epicycloid.

$$\beta'(t) = \left\langle \left(1 + \frac{1}{s}\right) \cos \frac{t}{s} - b \left(1 + \frac{1}{s}\right) \cos \left(\frac{t}{s} + t\right), - \left(1 + \frac{1}{s}\right) \sin \frac{t}{s} + b \left(1 + \frac{1}{s}\right) \sin \left(\frac{t}{s} + t\right) \right\rangle$$

and computing the slope, we find

$$- \frac{\left(1 + \frac{1}{s}\right) \cos \frac{t}{s} - b \left(1 + \frac{1}{s}\right) \cos \left(\frac{t}{s} + t\right)}{- \left(1 + \frac{1}{s}\right) \sin \frac{t}{s} + b \left(1 + \frac{1}{s}\right) \sin \left(\frac{t}{s} + t\right)}.$$

After canceling $\left(1 + \frac{1}{s}\right)$ and simplifying this expression is identical to (4).

Thus the slope of the line from the epicycloid to the corresponding tangent point is the same as the slope of the normal line at the same point on the epicycloid, so normal lines maintain the property of going through the point of tangency. \square

4 Distance Minimizing Paths

Classically, a minimal length curve connecting two points on a surface, S , must be a geodesic on S . In the classical sense a geodesic is a curve on S such that the acceleration is normal to S . Because of the presence of singularities on the surfaces we are studying, we are using the following alternate definition of geodesic.

Definition 7. A geodesic on a simple flat surface S is any curve corresponding with a straight line in a flat model for S .

Definition 8. A minimal geodesic connecting points A and B on a surface S is a geodesic on S at least as short as all other geodesics connecting A and B on S .

In this thesis we are seeking minimal geodesics on the surface of the conical cup and the tin can. In the case of the tubular side of the tin can it is clear that not all geodesics connecting two points are necessarily minimal geodesics since one helix may complete one circle around the can, but other helices connecting the same points may make two or more circles around the can.

4.1 Derivation of geodesics on cone

We wish to show that our definition of geodesic corresponds with the classical definition on the side of the cone. To find equations defining geodesics on the conical side of the cup, it is necessary to develop a parametrization mapping the conical side from the plane to the conical surface. We do this by thinking of wrapping a sector of a circle into the conical shape. The function $f(x, y)$ defined below will map a sector of a circle with edge along the x -axis with angle measured counter-clockwise from the x -axis into the conical surface in \mathbb{R}^3 . As is normal, α is the cone angle.

$$f(x, y) = \sqrt{x^2 + y^2} \sin \alpha \left\langle \cos \frac{\tan^{-1} \frac{y}{x}}{\sin \alpha}, \sin \frac{\tan^{-1} \frac{y}{x}}{\sin \alpha}, \cot \alpha \right\rangle$$

From classical theory, specifically, the Clairut Equations (found in [1]), we find the geodesics on the side of the cone can be found by solving the following differential equations:

$$x'' + \frac{E_x}{2E}x'^2 + \frac{E_y}{E}x'y' - \frac{G_x}{2E}y'^2 = 0 \quad (5)$$

and

$$y'' + \frac{E_y}{2G}x'^2 + \frac{G_x}{G}x'y' - \frac{G_y}{2G}y'^2 = 0 \quad (6)$$

where E and G are the coefficients from the first fundamental form,

$$\begin{aligned} E &= f_x \cdot f_x \\ E &= \frac{x^2 \cos^2 \alpha}{x^2 + y^2} + \\ &\left(\frac{x \sin \alpha \cos(\tan^{-1} \frac{y}{x} \csc \alpha)}{\sqrt{x^2 + y^2}} + \frac{y \sin(\tan^{-1} \frac{y}{x} \csc \alpha)}{\sqrt{x^2 + y^2}} \right)^2 + \\ &\left(\frac{x \sin \alpha \sin(\tan^{-1} \frac{y}{x} \csc \alpha)}{\sqrt{x^2 + y^2}} - \frac{y \cos(\tan^{-1} \frac{y}{x} \csc \alpha)}{\sqrt{x^2 + y^2}} \right)^2 \\ E &= 1 \end{aligned}$$

and analogously

$$G = f_y \cdot f_y = 1.$$

Substituting E and G into equations (5) and (6) yields the following simple ordinary differential equations:

$$x'' = 0 \rightarrow x = at + b$$

and

$$y'' = 0 \rightarrow y = ct + d.$$

It is obvious that these resulting parametric equations simply define a line in the plane. Thus straight lines in the domain set, a sector of a circle, map to geodesics on the cone.

4.2 Self Intersecting geodesics on the cone side

Ignoring the lid of the cup for a moment, let us consider an infinite cone. Unlike the helix on the side of a cylinder, a geodesic on the side of the cone may intersect with itself. In all cases, the criteria for a self-intersection are independent of distance of the geodesic from the cone vertex. Thus the number of self-intersections is a property of the cone - specifically, its cone angle. When the cone is unrolled the plane cone angle must be less than π for there to be a self-intersection.

Theorem 3. *Given a geodesic l on the side of an infinite circular cone with a plane cone angle of θ , the geodesic will self-intersect 1 less than the least integer greater than $\frac{\pi}{\theta}$ times, i.e., $\lceil \frac{\pi}{\theta} \rceil - 1$.*

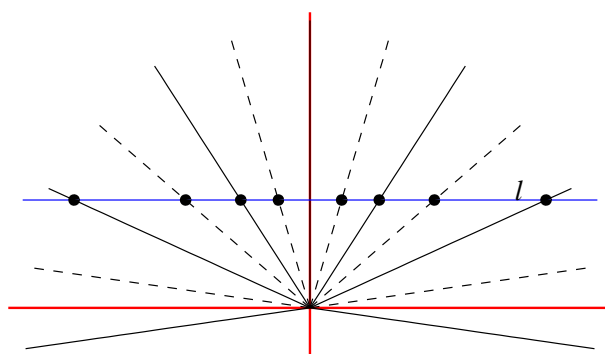


Figure 6: Multiple copies of an unrolled cone showing 4 self-intersections with a fifth self-intersection off to the side

Proof. As shown in Figure 6, unroll the cone into the plane by slicing on the radial line which cuts l closest to the vertex. Let h be the minimal distance from the geodesic to the vertex. Place the resulting sector of a circle in the first quadrant with the edge along the positive y -axis. When unrolled, l will be a straight line in the plane. We now place multiple copies on the xy plane with their sides connected such that l continues as a straight line through the copies.

Let points $A(x_a, y_a)$ and $B(x_b, y_b)$ be in the half-plane above the x -axis. For A and B to map to the same point on the cone and form a self-intersection, they must be on l and meet the following criteria:

- $\sqrt{x_a^2 + y_a^2} = \sqrt{x_b^2 + y_b^2}$
- $y_a = y_b = h$
- The radial lines connecting A and B to the origin must form an angle of $n\theta$ where $n \in \mathbb{N}$.

The above restraints imply

$$x_a^2 = x_b^2$$

and

$$x_a = -x_b,$$

which means there will be a self-intersection where A and B are reflections of each other across the y -axis. Thus, there will be one self-intersection for each ray forming an angle of $n\frac{\theta}{2}$ such that $0 < n\frac{\theta}{2} < \frac{\pi}{2}$, $n \in \mathbb{N}$. Thus the number of self intersections is

$$n = \left\lceil \frac{\pi}{\theta} \right\rceil - 1.$$

□

5 Geodesics over the Rim

To find minimal geodesics over the rim we will be looking at a flat model of the cup. We construct the flat model by thinking of cutting the conical side of the cup along some radial line and unrolling the side. We then flip up the lid and select an arbitrary point to attach the circular lid to the circular arc of the sector formed by unrolling the side of the cone. Realizing that we can pick any point on

the circular arc of the side to attach the lid, we imagine a copy of the lid at each of these points. In effect, this rolls the lid along the side. Suppose point B is on the lid of the cup and point A is on the side. When we roll the lid along the side, point B will trace out an epicycloid. To find a minimal geodesic, we find straight lines through point A that minimize distance to the epicycloid.

We know that a distance minimizing path from A to the epicycloid must be a straight line normal to the epicycloid. In Theorem 2, we showed that such a normal line must pass through the point of tangency P connecting the lid and the side. Thus we can simplify our search for a minimal geodesic by looking for a point of tangency P such that B, P and A are collinear.

In Figure 7, we see a flat model of a cup with a non-geodesic path connecting A and B . The following theorem states that we can find a flat model such that B, P and A are collinear making B - P - A a geodesic.

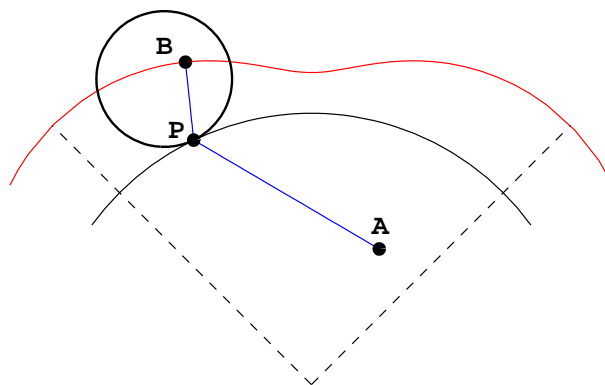


Figure 7: Epicycloid with path connecting A and B

Theorem 4. *Given point B on the lid of a cup and point A on the side, neither on the axis of revolution nor the rim, there exists a geodesic connecting B and A .*

Proof. We place the flat model of the cup so that the point corresponding to the vertex is at $(0, -s)$, where s is the slant height of the cone. Let b be the distance from point B to the center of the lid, let a be the distance from point A to the rim of the lid, and let s_0 be the cylindrical angle separating the two points. The epicycloid generated by point B is parametrized as before:

$$\beta(t) = \left\langle (1+s) \sin \frac{t}{s} - b \sin \left(\frac{t}{s} + t \right), -s + (1+s) \cos \frac{t}{s} - b \cos \left(\frac{t}{s} + t \right) \right\rangle.$$

The point of tangency P at time t is

$$\gamma(t) = \left\langle s \sin \frac{t}{s}, -s + s \cos \frac{t}{s} \right\rangle$$

and A is the point

$$\left((s-a) \sin \frac{s_0}{s}, -s + (s-a) \cos \frac{s_0}{s} \right).$$

We must show that there exists some t such that $\beta(t)$, $\gamma(t)$ and A are collinear. We can do this by thinking of \overrightarrow{BP} and \overrightarrow{AP} as vectors in \mathbb{R}^3 with a third component of 0 and computing their cross product.

$$\begin{aligned} \overrightarrow{BP} &= \left\langle (1+s) \sin \frac{t}{s} - b \sin \left(\frac{t}{s} + t \right) - s \sin \frac{t}{s}, -s + (1+s) \cos \frac{t}{s} - b \cos \left(\frac{t}{s} + t \right) + s - s \cos \frac{t}{s}, 0 \right\rangle \\ \overrightarrow{AP} &= \left\langle (s-a) \sin \frac{s_0}{s} - s \sin \frac{t}{s}, -s + (s-a) \cos \frac{s_0}{s} + s - s \cos \frac{t}{s}, 0 \right\rangle \end{aligned}$$

Now, computing and simplifying the cross product and letting $p(t)$ be the third component of $\overrightarrow{BP} \times \overrightarrow{AP}$:

$$p(t) = -(s-a) \sin \frac{s_0-t}{s} + bs \sin t + b(s-a) \sin \left(\frac{s_0-t}{s} - t \right). \quad (7)$$

Let us consider the function $p(t)$ for $t \in [0, s_0]$. Since $p(t)$ is continuous, we can use the Intermediate Value Theorem on this interval:

$$p(0) = -(1-b)(s-a) \sin \frac{s_0}{s} < 0$$

and

$$p(s_0) = ba \sin s_0 > 0.$$

Thus, there exists $t \in [0, s_0]$ such that $p(t) = 0$. This value of t indicates P such that B , P and A are collinear and thus there must be some geodesic from A to B . \square

The previous theorem indicates that there is a geodesic, but it does not indicate that it is unique. In some cases on the cup, there are up to three geodesics in this interval. The minimal geodesic can be determined through numerical methods.

We can find geodesics connecting a point on the lid of the tin can and a point on the side in the same way as we have just done on the cup. Alternatively, we can construct them by taking the limit as the cone height increases to infinity. To see this, we consider

$$\lim_{s \rightarrow \infty} \beta(t) = \lim_{s \rightarrow \infty} \left\langle (1+s) \sin \frac{t}{s} - b \sin \left(\frac{t}{s} + t \right), -s + (1+s) \cos \frac{t}{s} - b \cos \left(\frac{t}{s} + t \right) \right\rangle. \quad (8)$$

Computation of this limit is straightforward except for the term

$$\lim_{s \rightarrow \infty} s \sin \frac{t}{s}$$

We can compute this by letting $u = 1/s$ and then recognizing a more familiar limit

$$\lim_{u \rightarrow 0} \frac{\sin(ut)}{u} = t$$

Thus,

$$\lim_{s \rightarrow \infty} \beta(t) = \langle t - b \sin t, 1 - b \cos t \rangle.$$

The resulting expression is a parametrization for a cycloid. This makes sense since a cycloid would be generated by point B on the lid of the can if the lid were rolled along the straight rim of the can in the flat model.

In fact, if we take the $\lim_{s \rightarrow \infty} p(t)$, where $p(t)$ is defined in line (7) above, we find that

$$\begin{aligned} \lim_{s \rightarrow \infty} -(s-a) \sin \frac{s_0-t}{s} + bs \sin t + b(s-a) \sin \left(\frac{s_0-t}{s} - t \right) \\ = t - s_0 + b(s_0-t) \cos t + ab \sin t. \end{aligned}$$

This is exactly the expression that would be found if we constructed a geodesic over the rim of the tin can analogous to the preceding development on the cup.

6 Tin Can: Lid to Base

We will be working on a tin can with a point B on the lid of the can and a point A on the base of the can. Thus, we wish to determine geodesics of minimal length that cross both rims of the can and connect A and B .

First, let the lid with its rim be represented by a disk centered on the origin with a unit radius as in Figure 8. The side of the can of height h is then rolled around this lid keeping in contact at one tangent point as it rolls counter-clockwise around the lid. This way the tangent point is denoted by P at $(\cos t, \sin t)$. Thus on the rectangular side, the edge of the lid and the edge of the base map to two parallel lines with slope $\frac{-\cos t}{\sin t}$ which pass through points P and $((1+h) \cos t, (1+h) \sin t)$. In this way an equation for the line corresponding to the bottom rim can be found by:

$$\frac{-\cos t}{\sin t} = \frac{y - (1+h) \sin t}{x - (1+h) \cos t}$$

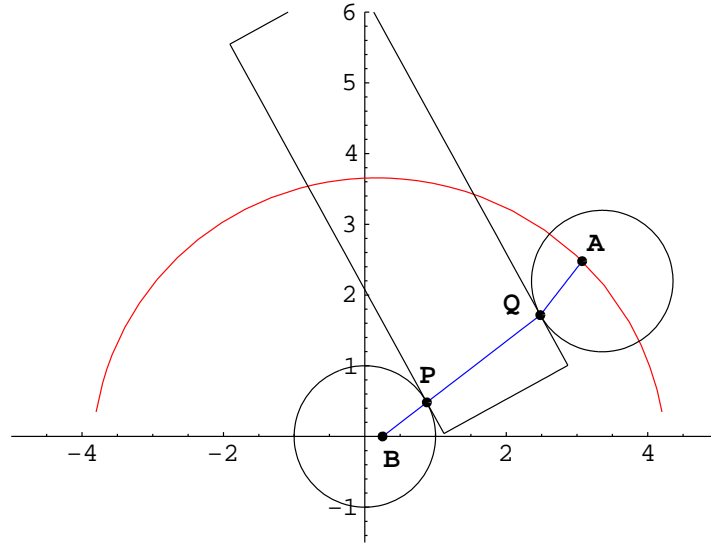


Figure 8: Unrolled can with centered lid

$$y = \frac{-x \cos t + (1+h) \cos t^2}{\sin t} + (1+h) \sin t$$

$$y = \frac{-x \cos t + 1 + h}{\sin t}. \quad (9)$$

Since a geodesic is any curve corresponding with a straight line in a flat model, we know the proposed geodesic must pass through B and P so that an equation for this line can also be found:

$$y = \frac{\sin t}{\cos t - r}(x - r) \quad (10)$$

We are now interested in the intersection of (9) and (10). This intersection can be found easily because they are simply lines in the plane. This intersection defines a point Q which is another point that the resulting linear geodesic passes through. This point is:

$$\left(\cos t + \frac{h(r - \cos t)}{r \cos t - 1}, \sin t - \frac{h \sin t}{r \cos t - 1} \right). \quad (11)$$

Next, it is helpful to think of the point at the intersection of the x -axis and the bottom rim (in the initial position) as an anchor point. The path of this point can be parametrized by

$$\langle (1+h) \cos t + t \sin t, (1+h) \sin t - t \cos t \rangle. \quad (12)$$

This parametrization is derived from the fact that the path is a roulette generated by rolling the straight edge of the side around the circular lid.

After Q is found, the center of the base is easily located at

$$\left\langle 2 \cos t + h \frac{r - \cos t}{r \cos t - 1}, 2 \sin t - h \frac{\sin t}{r \cos t - 1} \right\rangle. \quad (13)$$

Point Q must be the tangent point attaching the base of the can to the side. This is because the geodesic passes through Q so that if the side and base were not attached at Q the restraint that the geodesic be contained on the side of the can would be violated. By finding the distance to this point from the anchor point we can find out how far around the point A on the base must be rotated. This rotation parameter is affected by three things: First, the initial rotation s_0 is the angular displacement between A and B . Second, it is offset by the parameter t because the entire side of the can and correspondingly the base is rotated by this amount. And finally, there is an extra rotation resulting from the base being rolled from the “anchor” point to the point of tangency between the base and the side of the can.

This extra rotation is obviously the most complex of the three, but all that must be done is find the distance between the tangent point and the anchor point. This can be done with the simple distance formula or can be thought of as a vector found by subtracting the anchor vector (12) from the tangent point (11). The magnitude of this vector is the required distance. This subtraction when simplified results in:

$$\left\langle \frac{\sin t(t - rt \cos t + hr \sin t)}{r \cos t - 1}, \frac{-\cos t(t - rt \cos t + hr \sin t)}{r \cos t - 1} \right\rangle$$

We can factor out the common factor and then it is simple to find the magnitude:

$$\frac{t - rt \cos t + hr \sin t}{r \cos t - 1} \langle \sin t, -\cos t \rangle.$$

Since we are measuring distance we want the absolute value. Since

$$r < 1,$$

$$t - rt \cos t > 0, \text{ and}$$

$$r \cos t - 1 < 0,$$

and we look only at the range $0 < t < \pi$,

$$hr \sin t > 0,$$

the absolute value must negate the denominator and we can simplify to find the extra rotation:

$$s_0 = t + \frac{hr \sin t}{1 - r \cos t}. \quad (14)$$

Now, armed with (13) and (14), we can put all of the pieces together and obtain a parametrization for the path of point A on the base:

$$\left\langle 2 \cos t + h \frac{r - \cos t}{-1 + r \cos t} + q \cos \rho, 2 \sin t - h \frac{\sin t}{-1 + r \cos t} + q \sin \rho \right\rangle \quad (15)$$

where

$$\rho = s_0 + 2t + \frac{hr \sin t}{1 - r \cos t}.$$

We can now define vectors from point B to Q and Q to A . To find a minimal geodesic connecting B and A we must find a place where these vectors combine to form a straight line segment. We must locate the points P and Q where the four points, B , P , Q , and A are collinear. Analogous to the proof of Theorem 4, we can find this by thinking of \overrightarrow{BQ} and \overrightarrow{QA} as vectors in \mathbb{R}^3 , computing their cross product and finding where the third component of this product is 0.

7 Minimal Geodesics at the Vertex

The conical cup contains an interesting singularity not found on the surface of the can. This is the cone point. The following theorem gives us the basic result that is required to understand everything about minimal geodesics close to the cone point.

Theorem 5. *Every minimal geodesic from A to B on the side of the cup misses the cone point.*

Proof. Given any points A and B on the side of the cone, either A and B are diametrically opposed or they are not.

Suppose A and B are diametrically opposed. Cut the cone along any radial line not passing through A or B and unroll. Points A and B will be contained in a circular sector whose center corresponds to the cone point and whose angle is strictly less than π .

Suppose A and B are not diametrically opposed. We can construct a plane Q through the z -axis such that A and B are on the same side of Q . Cut the cone along one of the intersections of Q with

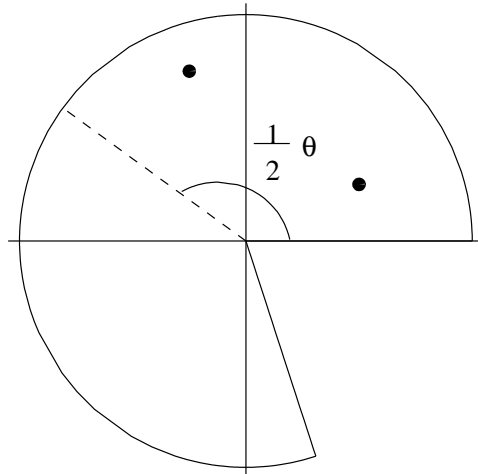


Figure 9: Construction of a minimal geodesic not passing through the cone vertex

the cone and unroll. Again, A and B will be contained in a circular sector centered at the cone vertex with angle strictly less than π .

In either case, A and B will be contained in a circular sector with angle less than a straight angle. Thus we can always construct a straight line through A and B , a minimal geodesic, that does not pass through the center of the sector, which corresponds to the cone point. \square

Note that in the special case when A and B are diametrically opposed there will be symmetric geodesics which may be minimal geodesics depending on the height of the cone.

8 Geodesics Minimizing Distance

As in the classical case, our definition of geodesic has a distance minimizing property.

Theorem 6. *Let A and B be any two points on a cup. Distance minimizing paths from A to B that cross the rim do so at most twice.*

Proof. Assume there exist points A and B that are connected by a minimal length path l that crosses the rim more than twice. Starting at A , travel along l until you arrive on the rim for the first time at point P . Similarly, starting at B , travel along l until you arrive on the rim for the first time at point Q . Then Q is distinct from P since l is a path from P to Q that passes through some point R on the rim distinct from P or Q . Thus there are points P, Q, R on the rim all on l . But by the

triangle inequality, the segment from P to Q on the lid is a shorter than the path from P to Q along l , contradicting the assumption that l was minimal. \square

Theorem 7. *Given two points A and B on the surface of the conical cup, every path from A to B of minimal length is a geodesic.*

Proof. There are four cases to consider.

Case 1: Suppose A and B are on the lid. The proof follows directly from the Hopf-Rinow Theorem (as stated in [1]) since our definition of geodesic corresponds to the classical definition on the lid.

Case 2: Suppose A and B are on the side connected by a minimal length path entirely on the side. The proof follows directly from the Hopf-Rinow Theorem since our definition of geodesic corresponds to the classical definition on the side.

Case 3: Choose a point A on the side of the cup and a point B on the top. Suppose A and B are connected by a non-geodesic path γ that intersects the interior of the lid non-trivially. We shall construct a path shorter than γ . By Theorem 6, the path γ intersects the rim at exactly one point P excluding B if it is on the rim. Now, consider a flat model for the cup as shown in Figure 10. By definition, the image of γ in the flat model is not straight. Nevertheless, by cases 1 and 2 above, γ is the union of two straight line segments β_1 connecting A to P and β_2 connecting P to B . Thus points B , P and A define a non-degenerate triangle in \mathbb{R}^2 . Although side AB of $\triangle APB$ is not a subset of the flat model, we can use it to construct a path that is shorter than γ . First note that $AP + PB > AB$, by the triangle inequality; we shall construct a path shorter than AB . In \mathbb{R}^2 , segment AB intersects the circle at point P' and intersects the arc of the sector at point P'' . Let $s_1 = P'P''$ and $s_2 = P'P''$. If $s_1 = s_2$, we can construct a shorter geodesic with the segments $\overline{BP'}$ and $\overline{P''A}$.

Suppose $s_1 < s_2$. Let γ be the epicycloid generated by P' as the circle rolls along the sector arc towards P'' . Let C be the first cusp that appears. We will show $P'P'' > CP''$. Let D be the intersection of $P'P''$ and the tangent line to the epicycloid at C . Construct the chord $\overline{CP''}$. Then $P'P'' > DP''$ since P' and P'' are on opposite sides of \overline{CD} . We have constructed $\triangle CDP''$ whose angle at C is obtuse since $\overline{CP''}$ is a chord of the sector arc and its cosine is negative. By the law of cosines,

$$(DP'')^2 = CD^2 + (CP'')^2 + 2|\cos\theta| > (CP'')^2.$$

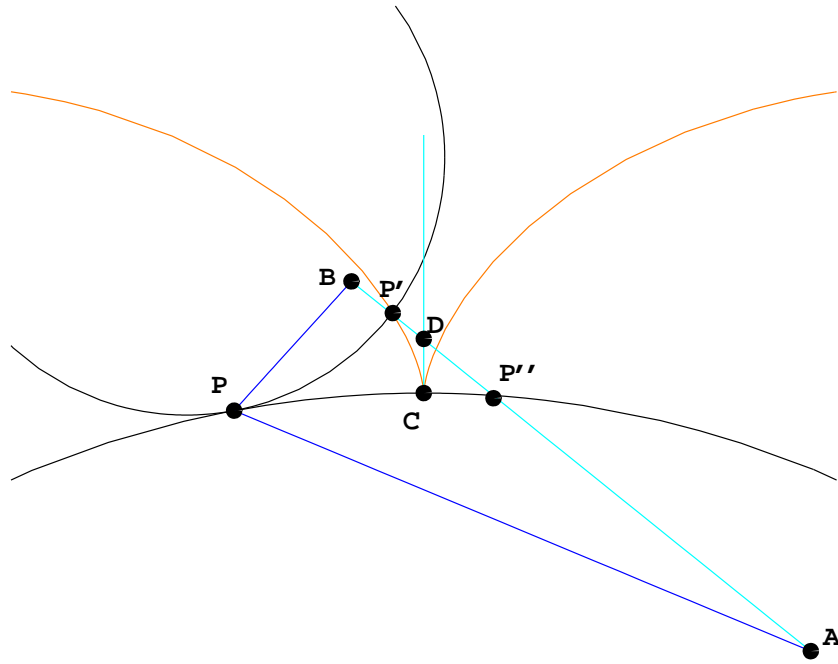


Figure 10: Shortening a non-geodesic path connecting A and B

Thus $DP'' > CP''$ and we have $P'P'' > DP'' > CP''$.

Suppose $s_2 > s_1$. The proof is similar except we let γ be the epicycloid generated by P'' as the arc rolls around the circle.

In both of these cases we can construct a shorter geodesic by joining segments $\overline{BP'}$, $\overline{P'P''}$ and $\overline{AP''}$.

Case 4: Suppose A and B are on the side connected by a minimal length path γ crossing the rim twice. Let P be the point of intersection between γ and the rim closest to A and Q be the analogous point closest to B . By the previous case, the shortest path from Q to A must be a geodesic and the shortest path from P to B must be a geodesic. Thus in the flat model, the segments \overline{AQ} and \overline{PB} must be straight sharing the segment \overline{QP} which means that the segment \overline{AB} must be straight. \square

Lemma 8. *Given two points A and B on the surface of a cup, not on the axis of revolution, that lie in the same half-plane of constant cylindrical angle θ_0 , there exists a unique minimal geodesic connecting them in the same half-plane of constant θ_0 .*

Proof. We consider three cases, each of which is easy to prove.

Case 1: Points A and B both lie on the lid of the cup

Since A and B are in the same cylindrical angle, they lie on the same radius of a circle in the flat model. The minimal geodesic must be the segment of the radius connecting them.

Case 2: Points A and B both lie on the side of the cup

Again, since A and B are in the same cylindrical angle, they lie on the same radius of a sector of a circle corresponding to the side of the cup in the flat model.

Case 3: Point A on the side and point B on the lid of the cup

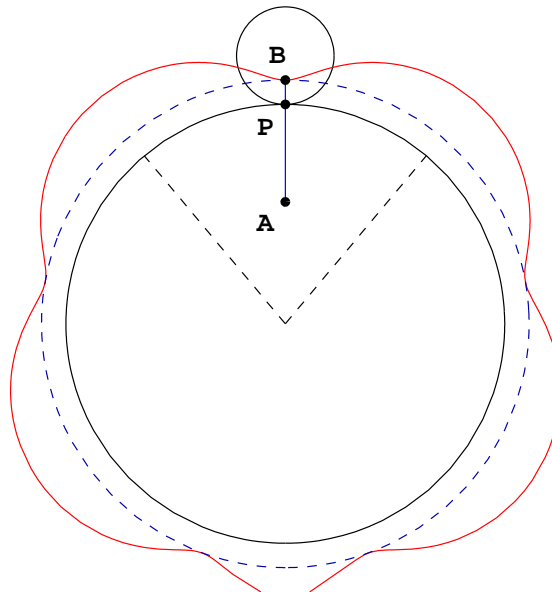


Figure 11: A unique minimal geodesic contained in a half-plane

This case is pictured in Figure 11 in which we have unrolled the cup and attached the lid at the unique point P lying on the rim and in the half-plane containing A and B . We have also extended the epicycloid and circular arc of the flat cone side to almost a full circle. Since A cannot be at the vertex of the sector of the circle by assumption, the dashed circle must be closest to A at point B . The epicycloid can never cross the dashed circle because B is at its closest point to the side of the cone by construction. Thus the closest the epicycloid ever gets to A is as it is pictured.

In all cases the unique minimal geodesic corresponds to a straight line entirely contained in the same plane that contains the A and B . □

Theorem 9. *Every minimal geodesic connecting two points A and B on the surface of the cup or tin can, not on the axis of revolution and not diametrically opposed, lies in the region of space*

between the two half-planes determined by A , B and the axis of revolution. Given two points A and B on the surface of a cup or tin can, not on the axis of revolution and not diametrically opposed, a minimal geodesic connecting A and B lies in the angle defined by the two planes containing A and B and the axis of revolution.

Proof. For contradiction, suppose A and B are connected by a minimal geodesic l that does not lie in the region of space between the half-planes of constant θ through A and B and the z -axis. By Lemma 8, l cannot cross any plane of constant angle θ more than once since a shorter geodesic could be constructed by connecting the two such points of intersection.

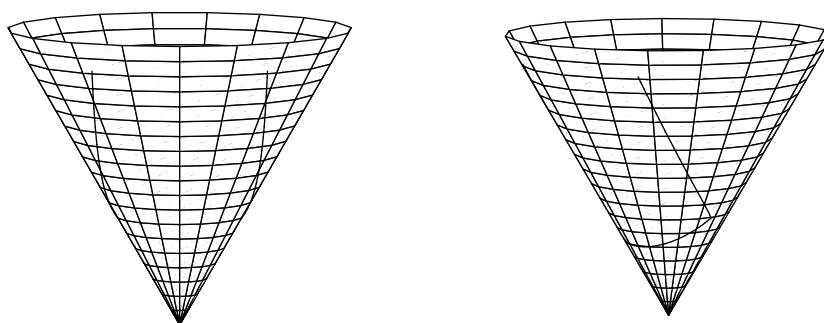


Figure 12: A geodesic spanning an angle more than π on the left-hand cone is reflected in a vertical plane and viewed on the right-hand cone, which has been rotated.

Consider a plane Q through the z -axis such that A and B are on the same side of Q . By Lemma 8, either l passes through Q twice or not at all. Suppose that l passes through Q twice. As shown in Figure 12, we then reflect the part of l opposite A and B across Q . This new path will not be a geodesic and can be shortened by applying Lemma 8. Thus l is not a minimal geodesic which means that any minimal geodesic must lie in the region of space indicated. \square

9 Non-Unique Minimal Geodesics

As implied in the previous section, there are cases when there are non-unique minimal geodesics connecting two points. One trivial example of this is the case when one point is the center of the lid of the conical cup and the other point is the cone point. In this case there is an infinite family of minimal geodesics. Such an analogous case also exists on the tin can.

9.1 Diametrically Opposed Points

In the case of the tin can, if we have points A and B diametrically opposed on the side of the can, we can find cases where there are four minimal geodesics. A flat model corresponding to such a case is shown in Figure 13 where there are two minimal geodesics going around the side, a minimal geodesic over the top, and one over the base. The magic height for such a tin can to exist with the shown geodesics is $\frac{\pi^2-4}{4}$ which will be found by taking the limit as the slant height increases to infinity of an analogous expression on the cone.

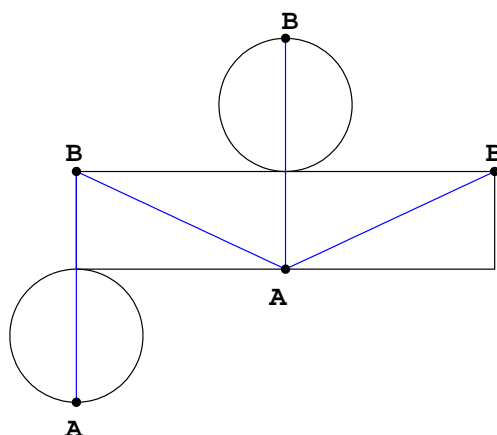


Figure 13: Four Minimal Geodesics connecting A and B

If the tin can in Figure 13 were taller, but point A and B remained the same distance from the lid, there would be only three minimal geodesics connecting A and B . We can generalize this case by considering diametrically opposed points A and B on the side of a cup.

Theorem 10. *Given a cup of sufficient height, there exist diametrically opposed points A and B such that there exist three minimal geodesics connecting them:*

- *one minimal geodesic over the top*
- *symmetric minimal geodesics around the side.*

Proof. Let B and A be diametrically opposed points on the side of a cup. Let b be the distance of B from the top of the cup. We can find the distance a from point A to the lid of the cup such that

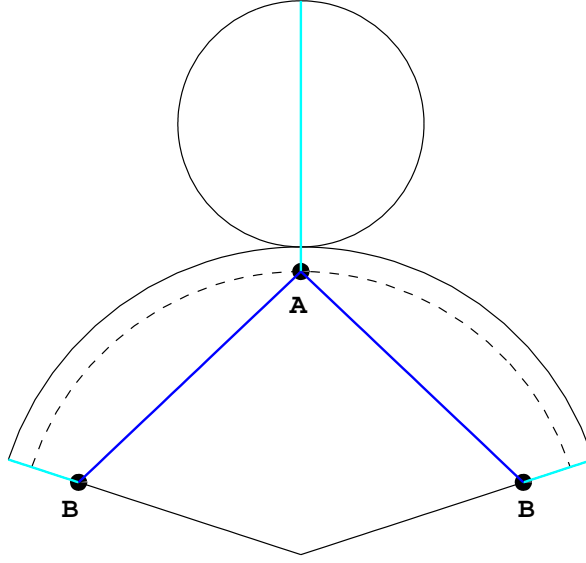


Figure 14: Three Minimal Geodesics connecting A and B

the required geodesics exist. The length of the geodesic over the top would be

$$L = 2 + a + b. \quad (16)$$

If we unroll the cup into a flat model, we can connect A and B with a straight line in a sector of a circle corresponding to the side of the cup. Supposing we are working on a unit radius cup, the plane cone angle in terms of the slant height will be $\frac{2\pi}{s}$. Thus in the flat model, since A and B are diametrically opposed, they form an angle of $\frac{\pi}{s}$ at the center of the sector. Thus, we can compute the length of a geodesic connecting A and B around the side of the cup using the law of cosines.

$$L^2 = (s - b)^2 + (s - a)^2 - 2(s - b)(s - a) \cos \frac{\pi}{r}. \quad (17)$$

Now, we can simply substitute (16) into (17) and we have an equation relating b and a :

$$(2 + a + b)^2 = (s - b)^2 + (s - a)^2 - 2(s - b)(s - a) \cos \frac{\pi}{r}.$$

Now solve for a and obtain

$$a = \frac{-2 + s^2 - 2b - bs + s(b - s) \cos(\pi/s)}{2 + b + s + (b - s) \cos(\pi/s)}. \quad (18)$$

So, by construction we can find a point A given an appropriate point B such that there exists three minimal geodesics. \square

Once again, we can recover an analogous result on the surface of the tin can by taking the limit of (18) as the cone height increases to infinity.

$$\lim_{s \rightarrow \infty} \frac{-2 + s^2 - 2b - bs + s(b-s) \cos(\pi/s)}{2 + b + s + (b-s) \cos(\pi/s)} = \frac{-4 - 4b + \pi^2}{4 + 4b} \quad (19)$$

To see that this is indeed the result on the tin can we only need to substitute 0 for b in (19) which becomes simply $\frac{\pi^2 - 4}{4}$ as above.

9.2 Open Questions Concerning Non-unique Minimal Paths

Suppose we are given point B on the lid of the cup, not in the center. Could we find a point A somewhere diametrically opposed to B on the side such that there are three minimal geodesics connecting A and B ? Analogously, on the tin can, is there a magic height such that there are four minimal geodesics connecting points B and A on the lid and base respectively?

There remain open questions about minimal geodesics in the general case when given points A and B are not diametrically opposed. As in the case when A and B are diametrically opposite each other, it may be that there exists a geodesic path crossing the rim twice of length equal to a geodesic path entirely contained on the side of the cup or tin can. It appears that given a point B close to the rim we could find a region R on the side of the cup where points in R are connected to B with a minimal geodesic over the top, but points not in R would be connected to B with a minimal geodesic around the side. On the boundary of R we may have non-unique minimal geodesics.

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