Geodesics on a Conical Drinking Cup With Lid

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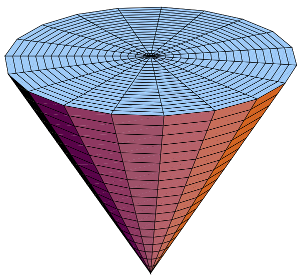
Abstract

There exist distance minimizing paths connecting two arbitrarily chosen points on the surface of a conical drinking cup with lid. These minimal length paths are geodesics, which can be found and analyzed by unrolling the cup and studying equivalent problems in the plane. General properties of geodesics on a cone will be discussed including results specific to geodesics near the cone point and geodesics connecting points near the rim.

Introduction to the Problem

The goal of this research is to characterize distance minimizing paths on the surface of a conical cup with a plane lid. Suppose that two points A and B are placed arbitrarily on a conical cup as described. What is the shortest route on the surface of the cup from point A to point B?

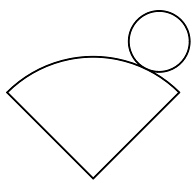


When the two points are on the conical part of the cup, the minimal paths on the conical side are often easy to find. Exceptions to this occur when the points are close to the rim of the cup.

Much is already known about minimal length paths on smooth general surfaces, but not around or over edges on surfaces. This research may ultimately help in finding minimal length paths (geodesics) over more general kinds of singularities.

Geodesics on the Cup

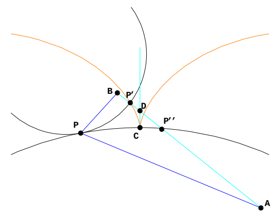
We can unroll the cone and form two distinct parts of the conical cup. The first part is the sector of a circle corresponding to the curved conical side of the cup. The second part is the circular lid of the cone. Such a flat model of the cone is shown in the picture below.



Classically, a minimal length curve connecting two points on a surface, S, must be a geodesic on S. In the classical sense a geodesic is a curve on S such that the acceleration is normal to S. Because of the presence of singularities on the surfaces we are studying, we define a geodesic to be any set of curves on the surface of the cone that map to straight lines in the plane when the cup is unrolled into a flat model.

As in the classical case, we wish to show that given any two points A and B on the surface of the conical cup, every path from A to B of minimal length is a geodesic. Let γ be a minimal length path from A to B. We show this with reference to four cases. In the first two cases, γ is entirely on the lid of the cup or entirely on the side of the cup. In both of these trivial cases, the result follows directly from the Hopf-Rinow Theorem of classical differential geometry.

In the third case, suppose that γ connects A on the side of the cup and a point B on the top. For contradiction, suppose the path γ is not a geodesic. We shall construct a path shorter than γ. We can trivially show that γ is not of minimum length unless it intersects the rim at exactly one point P excluding B if it is on the rim. Consider the part of a flat model for the cup as shown here.



By definition, the image of γ in the flat model is not straight. Nevertheless, by cases 1 and 2 above, γ is the union of two straight line segments β_1 connecting A to P and β_2 connecting P to B. Thus points B, P and A define a non-degenerate triangle in $\mathbb{R}^2$. Although side AB of $\triangle APB$ is not a subset of the flat model, we can use it to construct a path that is shorter than γ. First note that $AP + PB > AB$, by the triangle inequality; we shall construct a path shorter than AB. In $\mathbb{R}^2$, segment AB intersects the circle at point P' and intersects the arc of the sector at point P''. Let $s_1 = P'P''$ and $s_2 = \overline{P'P''}$. If $s_1 = s_2$, we can construct a shorter geodesic with the segments $\overline{BP'}$ and $\overline{P''A}$.

Suppose $s_1 < s_2$. Let γ be the epicycloid generated by P' as the circle rolls along the sector arc towards P''. Let C be the first cusp that appears. Let D be the intersection of $P'P''$ and the tangent line to the epicycloid at C. Since P' and P'' must be on opposite sides of the tangent line to the epicycloid at C, we have $P'P'' > DP''$ and by considering the law of cosines applied to $\triangle CDP''$ whose angle at C is obtuse $DP'' > CP''$. Taken together, $\overline{P'P''} > CP''$, and we can construct a path shorter than γ by connecting $\overline{BP'}$, $\overline{P'P''}$ and $\overline{AP''}$.

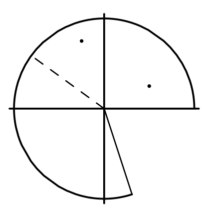
Suppose A and B are on the side connected by a minimal length path γ crossing the rim twice. Let P be the point of intersection between γ and the rim closest to A and Q be the analogous point closest to B. By the previous case, the shortest path from Q to A must be a geodesic and the shortest path from P to B must be a geodesic. Thus in the flat model, the segments $\overline{AQ}$ and $\overline{PB}$ must be straight sharing the segment $\overline{QP}$ which means that the segment $\overline{AB}$ must be straight.

Geodesics & the Cone Point

Given an infinite cone with points A and B somewhere on the cone, not on the point, a minimal geodesic connecting A and B will not pass through the cone point. Select a radial line on the side of the cone with the largest angle between the radial lines connecting A and B and cut along this line.

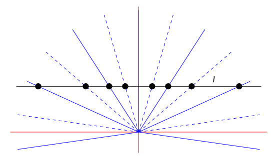
Points A and B must be contained in the interior of a semi-circle with center coincident with the cone point. Since a semi-circle is a convex region, the straight line connecting A and B corresponds to a geodesic on the side of the cone.

If a cone has a planar cone angle $\theta > \pi$, it is possible to construct geodesics that pass through the cone point. These geodesics are not the shortest ones on the conical surface to connect the points collinear with the cone point. Shorter geodesics can be found by a clever choice of a radial line on which to cut and unroll the cone. Such a choice forces A and B to lie in a sector of a circle which is strictly smaller than a semi-circle.



Self-Intersecting Geodesics

Geodesics on the cone surface will self intersect depending on the internal angle of the sector of the circle when unrolled; there will be $\left\lceil \frac{\pi}{\theta} \right\rceil - 1$ self-intersections when the lid of the cone is sufficiently far away from the cone point. The solid blue lines indicate multiple copies of an unrolled



conical surface. Note that the number of self-intersections is independent of the distance of the geodesic from the cone point.

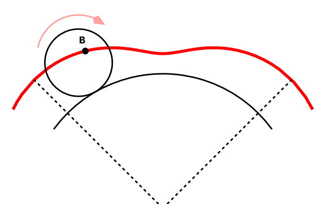
As shown in the picture above, unroll the cone into the plane by slicing on the radial line which cuts l closest to the cone point. Let h be the minimal distance from the geodesic to the cone point. Place the resulting sector of a circle in the first quadrant with the edge along the positive y-axis. When unrolled, l will be a straight line in the plane. We now place multiple copies on the xy plane with their sides connected such that l continues as a straight line through the copies.

Arbitrary points A and B will correspond to a self-intersection where A and B are reflections of each other across the y-axis. Thus, there will be one self-intersection for each ray forming an angle of $n\frac{\theta}{2}$ such that $0 < n\frac{\theta}{2} < \frac{\pi}{2}$, $n \in \mathbb{N}$. **Thus the geodesic will self-intersect 1 less than the least integer greater than $\frac{\pi}{\theta}$ times, i.e.,**

$$\left\lceil \frac{\pi}{\theta} \right\rceil - 1.$$

Geodesics Over the Rim

Suppose point B is on the lid of the cup and point A is on the conical side. If we roll the lid along the edge of the sector in the flat model of the cone, point B traces out an *epicycloid*. To find a minimal geodesic, we must find a straight line through point A that is normal (perpendicular) to the epicycloid and simultaneously passes through the tangent point.



It is easy to show that the normal line to the epicycloid always passes through the point of tangency between the two circles. We can parametrize this epicycloid by:

$$\beta(t) = \left\langle (a+b)\cos t - q\cos\left(\frac{a}{b}+1\right)t, (a+b)\sin t - q\sin\left(\frac{a}{b}+1\right)t \right\rangle$$

Calculate the slope of the line passing through a point on the epicycloid and the corresponding tangent point. This is:

$$\frac{(a+b)\sin t - q\sin\left(\frac{a}{b}+1\right)t - a\sin t}{(a+b)\cos t - q\cos\left(\frac{a}{b}+1\right)t - a\cos t}$$

or

$$\frac{b\sin t - q\sin\left(\frac{a}{b}+1\right)t}{b\cos t - q\cos\left(\frac{a}{b}+1\right)t} \qquad (1)$$

Calculate the slope of the normal line by taking the negative reciprocal of the slope of the line tangent to the epicycloid.

$$\beta'(t) = \left\langle -(a+b)\sin t + \left(\frac{a}{b}+1\right)q\sin\left(\frac{a}{b}+1\right)t, \right.$$
$$\left. (a+b)\cos t - \left(\frac{a}{b}+1\right)q\cos\left(\frac{a}{b}+1\right)t \right\rangle$$

Thus:

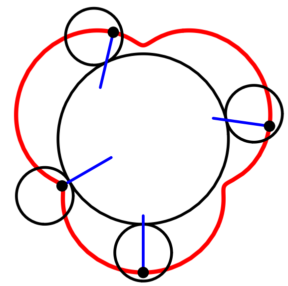
$$-\frac{-(a+b)\sin t + \left(\frac{a}{b}+1\right)q\sin\left(\frac{a}{b}+1\right)t}{(a+b)\cos t - \left(\frac{a}{b}+1\right)q\cos\left(\frac{a}{b}+1\right)t}$$

which can be simplified by multiplying $\frac{b}{b}$ and expanding:

$$\frac{ab\sin t + b^2\sin t - aq\sin\left(\frac{a}{b}+1\right)t - bq\sin\left(\frac{a}{b}+1\right)t}{ab\cos t + b^2\cos t - aq\cos\left(\frac{a}{b}+1\right)t - bq\cos\left(\frac{a}{b}+1\right)t}$$

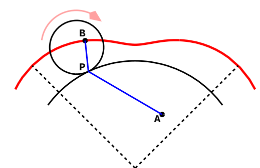
Simplify by grouping like terms and canceling; this is now identical to (1).

Since the slope of the line from the epicycloid to the corresponding tangent point is the same as the slope of the normal line at the same point on the epicycloid, we know that normal lines must maintain the property of passing through the corresponding point of tangency. This is illustrated by the figure below.



Minimizing Geodesics over the Rim

Consider a geodesic over the rim of the conical cup connecting a point on the top of the cup with a point over the edge on the conical side. An example of such a figure is shown below:



Apparently the path A-P-B in the figure is not of minimal length since it is not straight. However, it is not obvious that we should be able to find a straight that passes through any arbitrary point on the lid and the side.

With the help of the Intermediate Value Theorem, it is possible to prove that such a straight path is always constructible. We can show this by considering the cross product of two vectors thought of as vectors in space with a zero z-component. The first one is the vector connecting the point on the epicycloid with the tangent point of the two circular arcs remembering that this will be normal to the epicycloid:

$$\left\langle -r\cos t + (r+s)\cos t - (s-a)\cos\left(1+\frac{r}{s}\right)t, \right.$$
$$\left. -r\sin t + (r+s)\sin t - (s-a)\sin\left(1+\frac{r}{s}\right)t \right\rangle$$
$$\left\langle s\cos t - (s-a)\cos\left(1+\frac{r}{s}\right)t, s\sin t - (s-a)\sin\left(1+\frac{r}{s}\right)t \right\rangle \qquad (2)$$

The second is the vector connecting the tangent point and the point on the conical side of the cup:

$$\langle -b\cos s_0 + r\cos t, -b\sin s_0 + r\sin t \rangle \qquad (3)$$

The vectors in (2) and (3) will be parallel when their cross-product is zero. Computing the cross-product and simplifing yields:

$$p(t) = -bs\sin(s_0 - t) - (a-s)\left(r\sin\frac{rt}{s} + b\sin\left(s0 - \left(1+\frac{r}{s}\right)t\right)\right)$$

Obviously, p is a continuous function so we can apply the Intermediate Value Theorem on the interval $[0, s_0]$. For $t = 0$:

$$p(0) = -ab\sin s_0$$

and for $t = s_0$:

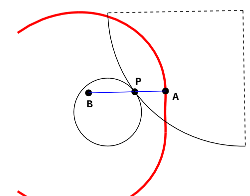
$$p(s_0) = (b-r)(a-s)\sin\frac{rs_0}{s}$$

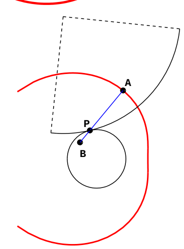
Since, $0 \le s_0 \le \pi$, $s > a > 0$, $r > b > 0$, and $s > r$ we conclude $p(0) \le 0$, and $p(s_0) \ge 0$. Thus, by the IVT, there exists t such that $p(t) = 0$. This value of t describes the desired geodesic by indicating the point on the rim where the path must cross so that it corresponds to a straight line in the flat model of the cone.

Uniqueness of Geodesics

Somewhat unexpectedly, geodesics connecting a point on the conical side and a point on top of the cup are not necessarily unique. It appears that there can be either 1, 2, or 3 geodesics connecting such points which meet basic criteria for being minimal. However, in most cases, one of these is indeed locally maximal while the other 2 are locally minimal with one of these 2 being shortest.

This leads to a case where only a slight movement of one of the points being connected results in a discontinuous jump of the minimal geodesic that connects them. This is shown in the following two figures showing the same cone with the difference that point A is 0.05 units further from the lid on the second one than on the first.





Open Question

Suppose that we are given a conical cup with a lid of unit radius. If we are trying to minimize distance between two points very close to the top rim of the cup, it may be shorter to go over the top than around the side of the cup.