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Some naturally occurring examples of A_∞-bialgebras

Ainhoa Berciano [a], Ronald Umble [b,*]

[a] Departamento de Didáctica de la Matemática y de las Ciencias Experimentales, Universidad del País Vasco, Bilbao, C.P. 48014, Spain

[b] Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551, United States

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ABSTRACT

Let p be an odd prime. We show that when $n \geq 3$, each tensor factor $E \otimes \Gamma$ of $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$ is an A_∞-bialgebra with non-trivial structure. We give explicit formulas for the structure maps and the relations among them. Thus $E \otimes \Gamma$ is a naturally occurring A_∞-bialgebra.

1. Introduction

Let p be an odd prime and let $n \geq 3$. Let $E(v, 2n+1)$ denote the $\mathbb{Z}_p$-exterior algebra on a generator v of dimension $2n+1$ and let $\Gamma(w, 2np+2)$ denote the $\mathbb{Z}_p$-divided power algebra on a generator w of dimension $2np+2$. In [2] and [3], Cartan, Eilenberg and Mac Lane showed that $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$ factors as an infinite tensor product with infinitely many tensor factors of the form $E \otimes \Gamma$; for example,

$$H_*(\mathbb{Z}, 3; \mathbb{Z}_p) \approx \bigotimes_{i\geq 0} E(v_i, 2p^i+1) \otimes \Gamma(w_i, 2p^{i+1}+2).$$

The main result in this paper is that for each $i \geq 0$, the factor

$$A_i = E(v_i, 2m+1) \otimes \Gamma(w_i, 2mp+2) \subset H_*(\mathbb{Z}, n; \mathbb{Z}_p)$$

is an A_∞-bialgebra with exactly three non-trivial "structurally compatible" operations, namely, a multiplication μ, a comultiplication Δ, and an operation $\Delta_p : A_i \to A_i^{\otimes p}$ of degree $p-2$. The operations Δ and Δ_p define the A_∞-coalgebra structure of A_i obtained by the first author using techniques of homological perturbation theory (see [1]), and can be realized as a contraction of the reduced bar construction $B(\mathbb{Z}[u]/(u^p))$ with $|u| = 2n$.

Since A_i is a Hopf algebra, Δ and μ are compatible in the sense that Δ is an algebra map, and it is natural to ask whether Δ_p and μ are compatible in some analogous way. Indeed,

$$f^p = \left(\Delta \otimes \mathbf{1}^{\otimes p-2}\right) \cdots (\Delta \otimes \mathbf{1})\, \Delta$$

is an algebra map, and Δ_p is compatible with μ as an "(f^p, f^p)-derivation of degree $p-2$". Consequently, A_i is an A_∞-bialgebra as defined by S. Saneblidze and the second author in [10]. We will refer to A_i as a "Hopf A_∞-coalgebra" to emphasize the compatibility of Δ and Δ_p with μ.

* Corresponding author. Tel.: +1 717 872 3531; fax: +1 717 871 2320.
E-mail address: Ron.Umble@millersville.edu (R. Umble).

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The paper is organized as follows: Section 2 reviews the notion of an A_∞-(co)algebra and the related tilde (co)bar construction. Section 3 reviews the A_∞-coalgebra structure on $E \otimes \Gamma$ mentioned above. Section 4 reviews the definition of the S-U diagonals Δ_P and Δ_K on the cellular chains of permutahedra and associahedra given in [9]. In Section 5 we give an exposition of the general notion of a "higher derivation", i.e., a Δ-derivation with respect to a Δ-compatible family of maps indexed by the faces of a family of polytopes $X = \sqcup_{n\geq 0} X_n$; the ideas in this section are implicit in [10]. We conclude the paper with Section 6, in which we prove Theorem 4: *Let p be an odd prime, let $i \geq 0$, and let $n \geq 1$. Then*

$$E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2)$$

is a Hopf A_∞-coalgebra over $\mathbb{Z}_p$.

2. A_∞-coalgebras and related constructions

Let R be a commutative ring with unity and let M be a graded R-module. Let $\overline{M} = M/M_0$, and let $\uparrow$ and $\downarrow$ denote the suspension and desuspension operators, which shift dimensions $+1$ and -1, respectively. Given a map $f : M \to N$ of graded modules, let $f_{i,j}$ denote the map

$$\mathbf{1}^{\otimes i} \otimes f \otimes \mathbf{1}^{\otimes j} : N^{\otimes i} \otimes M \otimes N^{\otimes j} \to N^{\otimes i+j+1}.$$

Given a connected differential graded (DG) algebra (A, d, μ), the *bar construction of A* is the tensor coalgebra $BA = T^c(\uparrow \bar{A})$ with cofree coproduct

$$\Delta_B[a_1|\cdots|a_n] = 1 \otimes [a_1|\cdots|a_n] + [a_1|\cdots|a_n] \otimes 1 + \sum_{i=1}^{n-1} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n]$$

and differential $d_B = d_t + d_a$, where

$$d_t = \sum_{i=1}^{n} (\uparrow d \downarrow)_{i-1,n-i} \quad \text{and} \quad d_a = \sum_{i=1}^{n-1} \left(\uparrow \mu \downarrow^{\otimes 2}\right)_{i-1,n-i-1}.$$

Dually, given a simply connected DG coalgebra (C, d, Δ), the *cobar construction of C* is the tensor algebra $\Omega C = T^a(\downarrow \bar{C})$ with differential $d_\Omega = d_t + d_c$, where

$$d_c = \sum_{i=1}^{n} \left(\downarrow^{\otimes 2} \Delta \uparrow\right)_{i-1,n-i}.$$

The notion of an A_∞-*algebra* was defined by J. Stasheff in his seminal paper [8]; here we review the dual notion of an A_∞-*coalgebra*. Given a simply connected graded R-module A together with a family of R-multilinear operations $\{\psi^k \in Hom^{k-2}\left(A, A^{\otimes k}\right)\}_{k\geq 1}$, let

$$d = \sum_{\substack{0\leq i\leq n-1;\ 1\leq k\leq n \\ n\geq 1}} \left(\downarrow^{\otimes k} \psi^k \uparrow\right)_{i,n-i-1} : \Omega A \to \Omega A.$$

Then $(A, \psi^n)_{n\geq 1}$ is an A_∞-*coalgebra* if $d \circ d = 0$, and the structure relations

$$\sum_{\substack{0\leq i\leq n-j-1 \\ 0\leq j\leq n-1;\ n\geq 1}} (-1)^{j(n+i+1)}\, \psi^{j+1}_{i,n-i-j-1} \psi^{n-j} = 0 \tag{2.1}$$

are given by the homogeneous components of the equation $d \circ d = 0$. The signs in (2.1) are the Koszul signs that appear when factoring out suspension and desuspension operators. Once the signs have been determined, we drop the simple-connectivity assumption and obtain the following abstract definition: An A_∞-*coalgebra* is a graded R-module A together with a family of multilinear operations

$$\left\{\psi^k \in Hom^{k-2}\left(A, A^{\otimes k}\right)\right\}_{k\geq 1}$$

that satisfy the structure relations in (2.1).

3. The A_∞-coalgebra $E \otimes \Gamma$

For $n \in \mathbb{N}$, let $Q_p(u, 2n) = \mathbb{Z}[u]/(u^p)$ with $|u| = 2n$. Recall that the exterior algebra $E(v, 2n-1)$ on a generator v of degree $2n-1$ is a Hopf algebra with primitively generated coproduct Δ. As a module, the divided power algebra $\Gamma(w, 2n)$

is generated by $\gamma_i = \gamma_i(w)_{i\geq 1}$, where $\gamma_1(w) = w$, and the algebra structure is defined by

$$\gamma_i\gamma_j = \frac{(i+j)!}{i!j!}\gamma_{i+j}.$$

Furthermore, $\Gamma(w, 2n)$ is a Hopf algebra with respect to the coproduct generated by

$$\Delta(\gamma_k(u)) = \sum_{i+j=k}\gamma_i(u)\otimes\gamma_j(u).$$

In [7], Prouté took the first steps toward computing the A_∞-coalgebra structure on $H = H_*\left(K(\pi, n); \mathbb{Z}_p\right)$. He showed that H is a classical coalgebra when $p = 2$. But when p is an odd prime, the A_∞-coalgebra structure explodes and he only obtained partial results in certain special cases. Thanks to Eilenberg and Mac Lane [3], there is a contraction (a special type of chain homotopy equivalence)

$$B(Q_p(u_i, 2np^i)) \to E(v_i, 2np^i + 1)\otimes\Gamma(w_i, 2np^{i+1}+2)$$

by which we obtain the decomposition

$$H_*(\mathbb{Z}, 3, \mathbb{Z}_p) \approx \bigotimes_{i\geq 0} E\left(w_i, 2p^i+1\right)\otimes\Gamma\left(x_i, 2p^{i+1}+2\right)$$

(see Theorem 4.5 in [1]). This idea extends inductively to $H_*(\mathbb{Z}, n, \mathbb{Z}_p)$ (see Theorem 4.24 of [1]). When $n = 4$, let $n_i = p^{i+1}+1$; then

$$H_*\left(\mathbb{Z}, 4; \mathbb{Z}_p\right) \approx \bigotimes_{i\geq 0}\left[\Gamma\left(y_i, 2p^i+2\right)\otimes\bigotimes_{j\geq 0} E\left(a_j, 2n_ip^j+1\right)\otimes\Gamma\left(b_j, 2n_ip^{j+1}+2\right)\right];$$

when $n = 5$ we have

$$H_*\left(\mathbb{Z}, 5; \mathbb{Z}_p\right) \approx \bigotimes_{i\geq 0}\left[\left(\bigotimes_{k\geq 0} E_{i,k}\otimes\Gamma_{i,k}\right)\otimes\bigotimes_{j\geq 0}\Gamma_{i,j}\otimes\left(\bigotimes_{l\geq 0} E_{i,j,l}\otimes\Gamma_{i,j,l}\right)\right];$$

and so on. In [1], the first author used these decompositions to extend Prouté's results and obtain

Theorem 1. *For $n \geq 3$, the induced A_∞-coalgebra operation Δ_q on $H_*(K(\mathbb{Z}, n); \mathbb{Z}_p)$ vanishes whenever $q \neq i(p-2)+2$ and $i \geq 0$.*

This result follows immediately from

Theorem 2. *For all $m \in \mathbb{N}$ and every odd prime p, the $\mathbb{Z}_p$-Hopf algebra*

$$A = E(v, 2m+1)\otimes\Gamma(w, 2mp+2)$$

is a non-trivial A_∞-coalgebra. The induced operation $\Delta_q : A \to A^{\otimes q}$ is non-trivial if and only if $q = 2$ or $q = p$. In fact, for $i = 0, 1$ and $\gamma_j = \gamma_j(w)$ we have

$$\Delta_2(v^i\gamma_j) = \sum_{k=0}^{i}\sum_{l=0}^{j} v^k\gamma_l\otimes v^{i-k}\gamma_{j-l}; \tag{3.1}$$

$$\Delta_p(v^i\gamma_j) = \sum_{k_1+\cdots+k_p=j-1} v^{i+1}\gamma_{k_1}\otimes\cdots\otimes v^{i+1}\gamma_{k_p}. \tag{3.2}$$

The coproduct Δ_2 defined in (3.1) is the induced coproduct on the tensor product of coalgebras and is compatible with the induced multiplication μ as an algebra map. And as we shall see, Δ_p is also compatible with μ as an "(f^p, f^p)-derivation of degree p".

4. The S-U diagonals on permutahedra and associahedra

This section gives a brief review of the S-U diagonals Δ_P and Δ_K on permutahedra $P = \sqcup_{n\geq 1}P_n$ and associahedra $K = \sqcup_{n\geq 2}K_n$ (up to sign); for details see [9]. Alternative constructions of Δ_K were subsequently given by Markl and Shnider [6] and Loday [5].

Let $n \in \mathbb{N}$ and let $\underline{n} = \{1, 2, \ldots, n\}$. A matrix E with entries from $\{0\} \cup \underline{n}$ is a *step matrix* if the following conditions hold:

- Each element of $\underline{n}$ appears as an entry of E exactly once.
- Elements of $\underline{n}$ in each row and column of E form an increasing contiguous block.
- Each diagonal parallel to the main diagonal of E contains exactly one element of $\underline{n}$.

The non-zero entries in a step matrix form a continuous staircase connecting the lower-left and upper-right entries. There is a bijective correspondence between step matrices with non-zero entries in $\underline{n}$ and permutations of $\underline{n}$.

Given a $q \times p$ integer matrix $M = (m_{ij})$, choose proper subsets $S_i \subset \{\text{non-zero entries in row } (i)\}$ and $T_j \subset \{\text{non-zero entries in col } (j)\}$, and define *down-shift* and *right-shift* operations D_{S_i} and R_{T_j} on M as follows:

(i) If $S_i \neq \varnothing$, $\max \text{row } (i+1) < \min S_i = m_{ij}$, and $m_{i+1,k} = 0$ for all $k \geq j$, then $D_{S_i}M$ is the matrix obtained from M by interchanging each $m_{ik} \in S_i$ with $m_{i+1,k}$; otherwise $D_{S_i}M = M$.
(ii) If $T_j \neq \varnothing$, $\max \text{col } (j+1) < \min T_j = m_{ij}$, and $m_{k,j+1} = 0$ for all $k \geq i$, then $R_{T_j}M$ is the matrix obtained from M by interchanging each $m_{k,j} \in T_j$ with $m_{k,j+1}$; otherwise $R_{Tj}M = M$.

Given a $q \times p$ step matrix E together with subsets $S_1, \ldots, S_q$ and $T_1, \ldots, T_p$ as above, there is the *derived matrix*

$$R_{T_p} \cdots R_{T_2} R_{T_1} D_{S_q} \cdots D_{S_2} D_{S_1} E.$$

In particular, step matrices are derived matrices under the trivial action with $S_i = T_j = \varnothing$ for all i, j.

Let $a = A_1|A_2| \cdots |A_p$ and $b = B_q|B_{q-1}| \cdots |B_1$ be partitions of $\underline{n}$. Then $a \times b$ is a (p, q)-*complementary pair* (CP) if there is a $q \times p$ derived matrix $M = \left(m_{ij}\right)$ such that $A_j = \{m_{ij} \neq 0 \mid 1 \leq i \leq q\}$ and $B_i = \{m_{ij} \neq 0 \mid 1 \leq j \leq p\}$. Thus (p, q)-CPs, which are in one-to-one correspondence with derived matrices, identify a particular set of product cells in $P_n \times P_n$.

Definition 1. Define

$$\Delta_P : C_0(P_1) \to C_0(P_1) \otimes C_0(P_1)$$

by $\Delta_P(\underline{1}) = \underline{1} \otimes \underline{1}$. Inductively, having defined

$$\Delta_P : C_*(P_k) \to C_*(P_k) \otimes C_*(P_k)$$

for all $k \leq n$, define Δ_P on $\underline{n+1} \in C_n(P_{n+1})$ by

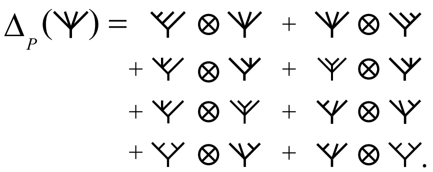
$$\Delta_P(\underline{n+1}) = \sum_{\substack{(p,q)\text{-CPs } a\times b \\ p+q=n+2}} \pm a \otimes b$$

and extend Δ_P to all of $C_*(P_{n+1})$ multiplicatively, i.e., define Δ_P on a generator $u_1| \cdots |u_r \in C_{n-r+1}(P_{n+1})$ by

$$\Delta_P(u_1| \cdots |u_r) = \Delta_P(u_1) | \cdots | \Delta_P(u_r).$$

Recall that faces of P_n in codimension k are indexed by planar-rooted trees with $n+1$ leaves and $k+1$ levels (PLTs).

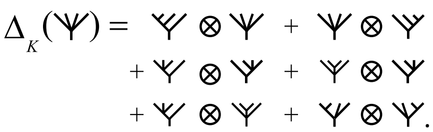
Example 1. In terms of PLTs, the diagonal Δ_P on the top cell of P_3 (up to sign) is given by



The diagonal Δ_p descends to a diagonal Δ_K on $C_*(K)$ via Tonks' cellular projection $\vartheta_0 : P_n \to K_{n+1}$, which forgets levels (see [12]). Faces of P_n indexed by PLTs with multiple nodes in the same level degenerate under ϑ_0, and corresponding generators span the kernel of the induced map $\vartheta_0 : C_*(P_n) \to C_*(K_{n+1})$. The diagonal Δ_K is given by

$$\Delta_K \vartheta_0 = (\vartheta_0 \otimes \vartheta_0)\Delta_P.$$

Example 2. When $n = 3$, the components $1|23 \otimes 13|2$ and $13|2 \otimes 3|12$ of $\Delta_P\left(\underline{3}\right)$ degenerate under ϑ because the tree corresponding to $13|2$ has two vertices in level 1; equivalently, $\dim(13|2) = 1$ whereas $\dim \vartheta(13|2) = 0$. Therefore (up to sign) the diagonal on the top cell of K_4 is given by



5. Δ-derivations and Δ_X-compatible families

Let $\{X_n\}_{n\geq 0}$ be a family of contractible polytopes such that $\dim X_n = n$, let $X = \sqcup_{n\geq 0} X_n$ and assume that the cellular chains $C_*(X)$ are equipped with a diagonal approximation

$$\Delta_X : C_*(X) \to C_*(X) \otimes C_*(X).$$

In this section we introduce the general notion of a Δ-derivation homotopy with respect to a Δ_X-compatible family of maps. When X_n is the n-simplex s_n, our definition agrees with the notion of a *high derivation* defined by Kadeishvili in [4]. When X_n is the n-dimensional permutahedron P_{n+1}, the notion of a Δ-derivation with respect to a Δ_P-compatible family is encoded in the construction of the biderivative given by S. Saneblidze and the second author (see [10,11,13]).

For each n, let n_k be the number of k-faces of X_n and choose a system of generators $\left\{x_i^k\right\}_{0\leq k\leq n;\, 1\leq i\leq n_k}$ for $C_*(X_n)$. Let X_i^k denote the smallest subcomplex of X_n containing the k-face associated with x_i^k. Given DGAs (A, μ_A, d_A) and (B, μ_B, d_B), let $\Theta : C_*(X_n) \to Hom(A, B)$ be a map of degree zero and let $\Theta_i^k = \Theta|_{C_*\left(X_i^k\right)}$.

Definition 2. The family of maps

$$\mathfrak{F}_n = \left\{\Theta\left(x_i^k\right)\right\}_{0\leq k<n;\, 1\leq i\leq n_k}$$

is Δ_X**-compatible** if each Θ_i^k is a chain map and the following diagram commutes:

$$\begin{array}{ccc}
C_k(X_i^k) & \xrightarrow{\Delta_X} & \sum_{p+q=k} C_p(X_i^k) \otimes C_q(X_i^k) \\
\Theta_i^k \downarrow & & \downarrow \Theta_i^k \otimes \Theta_i^k \\
Hom^k(A, B) & & \sum_{p+q=k} Hom^p(A, B) \otimes Hom^q(A, B) \\
(\mu_A)^* \downarrow & & \downarrow \approx \\
Hom^k(A \otimes A, B) & \xleftarrow[(\mu_B)_*]{} & Hom^k(A \otimes A, B \otimes B).
\end{array}$$

Let $\mathfrak{F}_n$ be a Δ_X-compatible family of maps. The map $T = \Theta\left(x_1^n\right) : A \to B$ associated with the top dimensional cell of X is a Δ**-derivation with respect to** $\mathfrak{F}_n$ if the diagram above commutes when $k = n$. If in addition, Θ is a chain map, then T is a Δ**-derivation homotopy with respect to** $\mathfrak{F}_n$. There is the dual notion of a Δ-coderivation homotopy with respect to a Δ_X-compatible family.

Let $\sigma_{n,2} : \left(A^{\otimes n}\right)^{\otimes 2} \to \left(A^{\otimes 2}\right)^{\otimes n}$ be the canonical permutation of tensor factors. Then for example,

$$\sigma_{3,2}\left(a_1|a_2|a_3 \otimes b_1|b_2|b_3\right) = (-1)^{|b_1||a_2|+|b_1||a_3|+|b_2||a_3|}\, a_1|b_1 \otimes a_2|b_2 \otimes a_3|b_3.$$

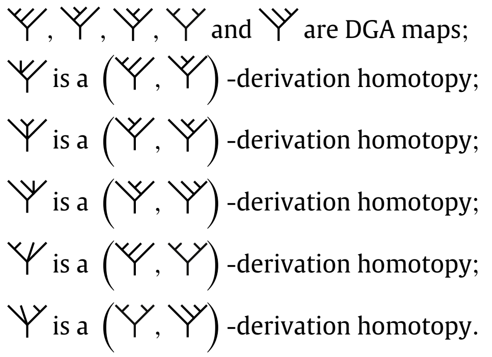
Example 3. Set $X_n = K_{n+2}$ and let Δ_K be the S-U diagonal on $C_*(K)$. Given a DGA (A, d, μ), let $d^{\otimes i}$ denote the free linear extension of d to $A^{\otimes i}$. Choose an arbitrary family of DG module maps $\{\Delta_i \in Hom^{i-2}\left(A, A^{\otimes i}\right)\}_{i\geq 2}$. For notational simplicity, identify Δ_i with the down-rooted i-leaf corolla and other down-rooted planar trees with their corresponding compositions in $Hom\left(A, A^{\otimes *}\right)$. When $n = 2$, K_2 is a point. If $\curlyvee$ is Δ-derivation homotopy with respect to the empty family $\mathfrak{F}_2$, then $\curlyvee$ is a DGA map, i.e.,

$$d^{\otimes 2}\curlyvee - \curlyvee d = 0 \quad \text{and} \quad \curlyvee\mu = \mu\, \sigma_{2,2}\left(\curlyvee \otimes \curlyvee\right).$$

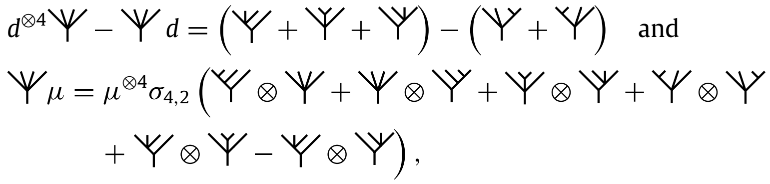
When $n = 3$, K_3 is an interval. If $\mathfrak{F}_3 = \left\{(\curlyvee\otimes 1)\curlyvee, (1\otimes\curlyvee)\curlyvee\right\}$ is a Δ_K-compatible family of compositions, then $(\curlyvee\otimes 1)\curlyvee$ and $(1\otimes\curlyvee)\curlyvee$ are DGA maps. If Ψ is a Δ-derivation homotopy with respect to $\mathfrak{F}_3$, then Ψ is a $\left((\curlyvee\otimes 1)\curlyvee, (1\otimes\curlyvee)\curlyvee\right)$-derivation homotopy, i.e.,

$$d^{\otimes 3}\,\Psi + \Psi d = (1\otimes\curlyvee)\curlyvee - (\curlyvee\otimes 1)\curlyvee \quad \text{and}$$
$$\Psi\mu = \mu^{\otimes 3}\, \sigma_{3,2}\left((\curlyvee\otimes 1)\curlyvee \otimes \Psi + \Psi \otimes (1\otimes\curlyvee)\curlyvee\right).$$

In the case of the pentagon K_4, assume that $\mathfrak{F}_4 = \left\{\Theta\left(x_i^k\right)\right\}_{k=0,1}$ is a Δ_K-compatible family. Then



If  is a Δ-derivation homotopy with respect to $\mathfrak{F}_4$, then



and so on.

Note that when the initial family of maps is $\{\Delta = \Delta_2, \Delta_n\}$, the only components of Δ_K that come into play are primitive and the Δ_K-compatible family $\mathfrak{F}_n$ is simply

$$\{f = (\Delta \otimes \mathbf{1}^{\otimes n-2}) \cdots (\Delta \otimes \mathbf{1})\Delta, g = (\mathbf{1}^{\otimes n-2} \otimes \Delta) \cdots (\mathbf{1} \otimes \Delta)\Delta\}.$$

We shall refer to a Δ-derivation with respect to such an $\mathfrak{F}_n$ as an "(f, g)-derivation of degree $n-2$".

Definition 3. Let (A, μ_A) and (B, μ_B) be graded algebras and let $f, g : A \to B$ be algebra maps. A map $h : A \to B$ of degree k is an (f, g)**-derivation of degree** k if

$$h\mu_A = \mu_B(f \otimes h + h \otimes g).$$

6. The Hopf A_∞-coalgebra $E \otimes \Gamma$

The natural A_∞-bialgebra structure on tensor factors $E \otimes \Gamma$ of $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$ is what we shall call a "Hopf A_∞-coalgebra". Let

$$f^n = \left(\Delta \otimes \mathbf{1}^{\otimes n-2}\right) \cdots (\Delta \otimes \mathbf{1})\, \Delta.$$

Definition 4. Let $n \geq 3$. A **Hopf** A_∞**-coalgebra** is an R-Hopf algebra (A, Δ, μ) together with an operation $\Delta_n \in Hom^{n-2}(A, A^{\otimes n})$ such that (A, Δ, Δ_n) is an A_∞-coalgebra and Δ_n is an (f^n, f^n)-derivation of degree $n-2$.

Our choice of terminology in Definition 4 is motivated by the fact that the operations $\{\Delta, \Delta_n, \mu\}$ satisfy the Hopf relation and its analogue in degree $n-2$, i.e,

$$\Delta_n\mu = \mu^{\otimes n}\sigma_{n,2}(f^n \otimes \Delta_n + \Delta_n \otimes f^n). \tag{6.1}$$

Our main result applies the following lemma, which follows from Vandermonde's identity:

Theorem 3 (*Vandermonde's Identity*)**.** *For $r, s \geq 0$ and $0 \leq k \leq r+s$,*

$$\binom{r+s}{k} = \sum_{i=0}^{k} \binom{r}{i}\binom{s}{k-i}.$$

Lemma 1. *Let $R = \mathbb{N} \cup \{0\}$ or $R = \mathbb{Z}_p$ with p prime. For all $i \geq 0$ and all n-tuples $(z_1, \ldots, z_n) \in R^n$ we have*

$$\binom{z_1 + \cdots + z_n + 1}{i} = \sum_{s_1+\cdots+s_n=i-1} \binom{z_1}{s_1} \cdots \binom{z_n}{s_n} + \sum_{t_1+\cdots+t_n=i} \binom{z_1}{t_1} \cdots \binom{z_n}{t_n},$$

where we reduce mod p *when $R = \mathbb{Z}_p$.*

Proof. A standard formula for binomial coefficients gives

$$\binom{z_1+\cdots+z_n+1}{i}=\binom{z_1+\cdots+z_n}{i-1}+\binom{z_1+\cdots+z_n}{i}.$$

Iteratively apply Vandermonde's identity to the right-hand summand and obtain

$$\binom{z_1+\cdots+z_n}{i}=\sum_{k_1=0}^{i}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{n-1}=0}^{k_{n-2}}\binom{z_1}{k_{n-1}}\binom{z_2}{k_{n-1}-k_{n-2}}\cdots\binom{z_n}{i-k_1}. \tag{6.2}$$

Note that the sum of the lower entries in the n binomial coefficients of this last expression is i and set

$$t_1=k_{n-1},\ t_2=k_{n-1}-k_{n-2},\ldots,t_{n-1}=k_1-k_2 \text{ and } t_n=i-k_1.$$

Then expression (6.2) can be rewritten as

$$\binom{z_1+\cdots+z_n}{i}=\sum_{t_1+\cdots+t_n=i}\binom{z_1}{t_1}\cdots\binom{z_n}{t_n}.\quad\square$$

Remark 1. The formula in Lemma 1 counts the number of ways i objects can be selected from a collection of $z_1+\cdots+z_n+1$ objects of $n+1$ different colors, one of which is uniquely colored " black" and z_i of which have the same unique color for each i. The first sum on the right-hand side counts the ways to select i objects one of which is black; the second sum counts the ways to select i objects none of which are black.

Theorem 4. *Let p be an odd prime. For each $i\geq 0$ and $n\geq 1$, let*

$$A_i=E(v_i,2np^i+1)\otimes\Gamma(w_i,2np^{i+1}+2),$$

let $\mu=(\mu_E\otimes\mu_\Gamma)\sigma_{2,2}$, and for $j=2,p$, let Δ_j be defined as in (3.1) and (3.2). Then $(A_i,\Delta_2,\Delta_p,\mu)$ is a Hopf A_∞-coalgebra over $\mathbb{Z}_p$.

Proof. For each $n\geq 1$ and $i\geq 0$, (A_i,Δ_2,μ) is a Hopf algebra via $\mu=(\mu_E\otimes\mu_\Gamma)\sigma_{2,2}$, where

$$\mu_E(v_i\otimes v_i)=0\quad\text{and}\quad \mu_\Gamma(\gamma_i(w_i)\otimes\gamma_j(w_i))=\binom{i+j}{i}\gamma_{i+j}(w_i),$$

and (A_i,Δ_2,Δ_p) is an A_∞-coalgebra by Theorem 2. Furthermore, both sides of relation (6.1) vanish on tensor products involving v_i, and it is sufficient to evaluate this relation on $\gamma_i\otimes\gamma_j=\gamma_i(w_i)\otimes\gamma_j(w_i)$. First,

$$\begin{aligned}\Delta_p\mu\left(\gamma_i\otimes\gamma_j\right)&=\binom{i+j}{i}\Delta_p(\gamma_{i+j})\\&=\sum_{z_1+\cdots+z_p=i+j-1}\binom{i+j}{i}v_i\gamma_{z_1}\otimes\cdots\otimes v_i\gamma_{z_p}\\&=\sum_{z_1+\cdots+z_p=i+j-1}\binom{z_1+\cdots+z_p+1}{i}u,\end{aligned}\tag{6.3}$$

where $u=v_i\gamma_{z_1}\otimes\cdots\otimes v_i\gamma_{z_p}$, and second,

$$\begin{aligned}&\mu^{\otimes p}\sigma_{p,2}\left(f^p\otimes\Delta_p+\Delta_p\otimes f^p\right)\left(\gamma_i\otimes\gamma_j\right)\\&=\sum_{\substack{l_1+\cdots+l_p=j\\ s_1+\cdots+s_p=i-1}}\binom{l_1+s_1}{s_1}\cdots\binom{l_p+s_p}{s_p}v_i\gamma_{l_1+s_1}\otimes\cdots\otimes v_i\gamma_{l_p+s_p}\\&\quad+\sum_{\substack{m_1+\cdots+m_p=j-1\\ t_1+\cdots+t_p=i}}\binom{m_1+t_1}{t_1}\cdots\binom{m_p+t_p}{t_p}v_i\gamma_{m_1+t_1}\otimes\cdots\otimes v_i\gamma_{m_p+t_p}\\&=\sum_{z_1+\cdots+z_p=i+j-1}\left[\sum_{s_1+\cdots+s_p=i-1}\binom{z_1}{s_1}\cdots\binom{z_p}{s_p}+\sum_{t_1+\cdots+t_p=i}\binom{z_1}{t_1}\cdots\binom{z_p}{t_p}\right]u.\end{aligned}\tag{6.4}$$

But expressions (6.3) and (6.4) are equal modulo p by Lemma 1. $\square$

In view of Eilenberg and Mac Lane's decomposition of $H_*(\mathbb{Z},n;\mathbb{Z}_p)$ discussed in Section 3, the submodule

$$A_i=E(v_i,2np^i+1)\otimes\Gamma(w_i,2np^{i+1}+2)\subset H_*\left(\mathbb{Z},n;\mathbb{Z}_p\right)$$

is a naturally occurring Hopf A_∞-coalgebra for each $i\geq 0$. Furthermore, when $n=3$, the Hopf A_∞-coalgebra structures on the A_i's induce a global A_∞-bialgebra structure on $H_*\left(\mathbb{Z},3;\mathbb{Z}_p\right)$ as a formal (as yet undefined) tensor product of A_∞-bialgebras. The relationship between this global structure and the underlying topology of $K\left(\mathbb{Z},3\right)$ is an interesting open question.

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