

Some naturally occurring examples of $A_{\infty}$-bialgebras

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A R T I C L E  I N F O

Article history:
Received 25 June 2007
Received in revised form 27 December 2009
Available online 15 September 2010
Communicated by J. Huebschmann

M S C : 18D50 (operads); 55P20 (Eilenberg–Mac Lane spaces); 55P43 (spectra with additional structure);
57T30 (bar and cobar constructions)

A B S T R A C T

Let $p$ be an odd prime. We show that when $n \geq 3$, each tensor factor $E \otimes \Gamma$ of $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$ is an $A_{\infty}$-bialgebra with non-trivial structure. We give explicit formulas for the structure maps and the relations among them. Thus $E \otimes \Gamma$ is a naturally occurring $A_{\infty}$-bialgebra.

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1. Introduction

Let $p$ be an odd prime and let $n \geq 3$. Let $E(v, 2n + 1)$ denote the $\mathbb{Z}_p$-exterior algebra on a generator $v$ of dimension $2n + 1$ and let $\Gamma(w, 2np + 2)$ denote the $\mathbb{Z}_p$-divided power algebra on a generator $w$ of dimension $2np + 2$. In [2] and [3], Cartan, Eilenberg and Mac Lane showed that $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$ factors as an infinite tensor product with infinitely many tensor factors of the form $E \otimes \Gamma$; for example,

$$H_*(\mathbb{Z}, 3; \mathbb{Z}_p) \approx \bigotimes_{i=0} E(v_i, 2p^i + 1) \otimes \Gamma(w_i, 2p^{i+1} + 2).$$

The main result in this paper is that for each $i \geq 0$, the factor

$$A_i = E(v_i, 2m + 1) \otimes \Gamma(w_i, 2mp + 2) \subset H_*(\mathbb{Z}, n; \mathbb{Z}_p)$$

is an $A_{\infty}$-bialgebra with exactly three non-trivial "structurally compatible" operations, namely, a multiplication $\mu$, a comultiplication $\Delta$, and an operation $\Delta_p : A_i \to A_i^{\otimes p}$ of degree $p - 2$. The operations $\Delta$ and $\Delta_p$ define the $A_{\infty}$-coalgebra structure of $A_i$ obtained by the first author using techniques of homological perturbation theory (see [1]), and can be realized as a contraction of the reduced bar construction $B(\mathbb{Z}[u]/(u^p))$ with $|u| = 2n$.

Since $A_i$ is a Hopf algebra, $\Delta$ and $\mu$ are compatible in the sense that $\Delta$ is an algebra map, and it is natural to ask whether $\Delta_p$ and $\mu$ are compatible in some analogous way. Indeed,

$$f^p = (\Delta \otimes 1^{\otimes p-2}) \cdots (\Delta \otimes 1) \Delta$$

is an algebra map, and $\Delta_p$ is compatible with $\mu$ as an "$(f^p, f^p)$-derivation of degree $p-2$". Consequently, $A_i$ is an $A_{\infty}$-bialgebra as defined by S. Saneblidze and the second author in [10]. We will refer to $A_i$ as a "Hopf $A_{\infty}$-coalgebra" to emphasize the compatibility of $\Delta$ and $\Delta_p$ with $\mu$. 

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doi:10.1016/j.jpaa.2010.08.012
The paper is organized as follows: Section 2 reviews the notion of an $A_\infty$-(co)algebra and the related tilde (co)bar construction. Section 3 reviews the $A_\infty$-coalgebra structure on $E \otimes \Gamma$ mentioned above. Section 4 reviews the definition of the $S$-U diagonals $\Delta_p$ and $\Delta_Q$ on the cellular chains of permutahedra and associahedra given in [9]. In Section 5 we give an exposition of the general notion of a “higher derivation”, i.e., a $\Delta$-derivation with respect to a $\Delta$-compatible family of maps indexed by the faces of a family of polytopes $X = \biguplus_{n \geq 0} X_n$; the ideas in this section are implicit in [10]. We conclude the paper with Section 6, in which we prove Theorem 4: Let $p$ be an odd prime, let $i \geq 0$, and let $n \geq 1$. Then

$$E(u_i, 2np^i + 1) \otimes \Gamma (u_i, 2np^{i+1} + 2)$$

is a Hopf $A_\infty$-coalgebra over $\mathbb{Z}_p$.

2. $A_\infty$-coalgebras and related constructions

Let $R$ be a commutative ring with unity and let $M$ be a graded $R$-module. Let $M = M/M_0$, and let $\uparrow$ and $\downarrow$ denote the suspension and desuspension operators, which shift dimensions $+1$ and $-1$, respectively. Given a map $f : M \to N$ of graded modules, let $f_{i,j}$ denote the map

$$1 \otimes f \otimes 1 : N^{\otimes i} \otimes M \otimes N^{\otimes j} \to N^{\otimes i+j+1}.$$ 

Given a connected differential graded (DG) algebra $(A, d, \mu)$, the bar construction of $A$ is the tensor coalgebra $BA = T^d(\uparrow A)$ with cofree coproduct

$$\Delta_A[a_1 \cdots a_n] = 1 \otimes [a_1 \cdots a_n] + [a_1 \cdots a_n] \otimes 1 + \sum_{i=1}^{n-1} [a_1 \cdots [a_i \cdots a_n] \otimes [a_{i+1} \cdots a_n]$$

and differential $d_B = d_t + d_a$, where

$$d_t = \sum_{i=1}^{n} (\uparrow \downarrow)_{i-1,n-i} \quad \text{and} \quad d_a = \sum_{i=1}^{n-1} (\uparrow \downarrow \uparrow)_{i-1,n-i-1}.$$ 

Dually, given a simply connected DG coalgebra $(C, d, \Delta)$, the cobar construction of $C$ is the tensor algebra $\Omega C = T^d(\downarrow \bar{C})$ with differential $d_\Omega = d_t + d_c$, where

$$d_c = \sum_{i=1}^{n} (\downarrow \Delta \uparrow)_{i-1,n-i}.$$ 

The notion of an $A_\infty$-algebra was defined by J. Stasheff in his seminal paper [8]; here we review the dual notion of an $A_\infty$-coalgebra. Given a simply connected graded $R$-module $A$ together with a family of $R$-multilinear operations $\{ \psi^k \in \text{Hom}^{k-2}(A, A^{\otimes k}) \}_{k \geq 1}$, let

$$d = \sum_{0 \leq l \leq n-1; 1 \leq k \leq n \atop n \geq 1} \psi^k \uparrow_{i,n-i-1} : \Omega A \to \Omega A.$$ 

Then $(A, \psi^n)_{n \geq 1}$ is an $A_\infty$-coalgebra if $d \circ d = 0$, and the structure relations

$$\sum_{0 \leq i \leq n-j-1; 0 \leq j \leq n-1 \atop n \geq 1} (-1)^{i(n+i+1)} \psi^{i+1}_{i,n-i-j-1} \psi^{n-j} = 0 \quad (2.1)$$

are given by the homogeneous components of the equation $d \circ d = 0$. The signs in (2.1) are the Koszul signs that appear when factoring out suspension and desuspension operators. Once the signs have been determined, we drop the simple-connectivity assumption and obtain the following abstract definition: An $A_\infty$-coalgebra is a graded $R$-module $A$ together with a family of multilinear operations

$$\{ \psi^k \in \text{Hom}^{k-2}(A, A^{\otimes k}) \}_{k \geq 1}$$

that satisfy the structure relations in (2.1).

3. The $A_\infty$-coalgebra $E \otimes \Gamma$

For $n \in \mathbb{N}$, let $Q_p (u, 2n) = \mathbb{Z}[u]/(u^p)$ with $|u| = 2n$. Recall that the exterior algebra $E(v, 2n-1)$ on a generator $v$ of degree $2n-1$ is a Hopf algebra with primitively generated coproduct $\Delta$. As a module, the divided power algebra $\Gamma (u, 2n)$
is generated by \( \gamma_1 = \gamma_1(w)_{i \geq 1} \), where \( \gamma_1(w) = w \), and the algebra structure is defined by
\[
\gamma \gamma_i = \frac{(i + j)!}{i!j!} \gamma_{i+j}.
\]
Furthermore, \( \Gamma'(w, 2n) \) is a Hopf algebra with respect to the coproduct generated by
\[
\Delta(\gamma_k(u)) = \sum_{i+j=k} \gamma_i(u) \otimes \gamma_j(u).
\]

In [7], Prouté took the first steps toward computing the \( A_\infty \)-coalgebra structure on \( H = H_n(K(\pi, n); \mathbb{Z}_p) \). He showed that \( H \) is a classical coalgebra when \( p = 2 \). But when \( p \) is an odd prime, the \( A_\infty \)-coalgebra structure explodes and he only obtained partial results in certain special cases. Thanks to Eilenberg and Mac Lane [3], there is a contraction (a special type of chain homotopy equivalence)
\[
B(Q, 2np^i) \to E(w_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2)
\]
by which we obtain the decomposition
\[
H_*(\mathbb{Z}, 3; \mathbb{Z}_p) \cong \bigotimes_{i \geq 0} E(w, 2p^i + 1) \otimes \Gamma(x_i, 2p^{i+1} + 2)
\]
(see Theorem 4.5 in [1]). This idea extends inductively to \( H_*(\mathbb{Z}, n; \mathbb{Z}_p) \) (see Theorem 4.24 of [1]). When \( n = 4 \), let \( n_i = p^{i+1} + 1 \); then
\[
H_*(\mathbb{Z}, 4; \mathbb{Z}_p) \cong \bigotimes_{i \geq 0} \left[ \Gamma(y_i, 2p^i + 2) \otimes \bigotimes_{j \geq 0} \left( E(a_j, 2np^j + 1) \otimes \Gamma(b_j, 2np^{j+1} + 2) \right) \right]
\]
when \( n = 5 \) we have
\[
H_*(\mathbb{Z}, 5; \mathbb{Z}_p) \cong \bigotimes_{i \geq 0} \left[ \left( \bigotimes_{k \geq 0} E_{i,k} \otimes \Gamma_{i,k} \right) \otimes \bigotimes_{j \geq 0} \left( \bigotimes_{l \geq 0} E_{i,j,l} \otimes \Gamma_{i,j,l} \right) \right]
\]
and so on. In [1], the first author used these decompositions to extend Prouté’s results and obtain

**Theorem 1.** For \( n \geq 3 \), the induced \( A_\infty \)-coalgebra operation \( \Delta_q \) on \( H_*(K(\mathbb{Z}, n); \mathbb{Z}_p) \) vanishes whenever \( q \neq i(p - 2) + 2 \) and \( i \geq 0 \).

This result follows immediately from

**Theorem 2.** For all \( m \in \mathbb{N} \) and every odd prime \( p \), the \( \mathbb{Z}_p \)-Hopf algebra
\[
A = E(v, 2m + 1) \otimes \Gamma(w, 2mp + 2)
\]
is a non-trivial \( A_\infty \)-coalgebra. The induced operation \( \Delta_q : A \to A^{\otimes q} \) is non-trivial if and only if \( q = 2 \) or \( q = p \). In fact, for \( i = 0, 1 \) and \( \gamma_j = \gamma_j(w) \) we have
\[
\Delta_2(v^i \gamma_j) = \sum_{k=0}^{i} \sum_{l=0}^{j} v^k \gamma_l \otimes v^{i-k} \gamma_j-l; \quad (3.1)
\]
\[
\Delta_p(v^i \gamma_j) = \sum_{k_1+\cdots+k_p=j-1} v^{i+1} \gamma_k_1 \otimes \cdots \otimes v^{i+1} \gamma_k_p. \quad (3.2)
\]

The coproduct \( \Delta_2 \) defined in (3.1) is the induced coproduct on the tensor product of coalgebras and is compatible with the induced multiplication \( \mu \) as an algebra map. And as we shall see, \( \Delta_p \) is also compatible with \( \mu \) as an “\((f^p, f^p)\)-derivation of degree \( p \)”.

4. The S-U diagonals on permutahedra and associahedra

This section gives a brief review of the S-U diagonals \( \Delta_p \) and \( \Delta_K \) on permutahedra \( P = \sqcup_{n \geq 1} P_n \) and associahedra \( K = \sqcup_{n \geq 2} K_n \) (up to sign); for details see [9]. Alternative constructions of \( \Delta_K \) were subsequently given by Markl and Shnider [6] and Loday [5].
Let \( n \in \mathbb{N} \) and let \( n = \{1, 2, \ldots, n\} \). A matrix \( E \) with entries from \([0] \cup n\) is a step matrix if the following conditions hold:

- Each element of \( n \) appears as an entry of \( E \) exactly once.
- Elements of \( n \) in each row and column of \( E \) form an increasing contiguous block.
- Each diagonal parallel to the main diagonal of \( E \) contains exactly one element of \( n \).

The non-zero entries in a step matrix form a continuous staircase connecting the lower-left and upper-right entries. There is a bijective correspondence between step matrices with non-zero entries in \( n \) and permutations of \( n \).

Given a \( q \times p \) integer matrix \( M = (m_{ij}) \), choose proper subsets \( S_i \subset \{\text{non-zero entries in row } (i)\} \) and \( T_j \subset \{\text{non-zero entries in col } (j)\} \), and define down-shift and right-shift operations \( D_S \) and \( R_T \) on \( M \) as follows:

1. If \( S_i \neq \emptyset \), max row \((i + 1) < \min S_i = m_{ij} \), and \( m_{i,k} = 0 \) for all \( k \geq j \), then \( D_S M \) is the matrix obtained from \( M \) by interchanging each \( m_{jk} \in S_i \) with \( m_{i,k} \); otherwise \( D_S M = M \).
2. If \( T_j \neq \emptyset \), max col \((j + 1) < \min T_j = m_{ij} \), and \( m_{k,j} = 0 \) for all \( k \geq i \), then \( R_T M \) is the matrix obtained from \( M \) by interchanging each \( m_{ij} \in T_j \) with \( m_{k,j} \); otherwise \( R_T M = M \).

Given a \( q \times p \) step matrix \( E \) together with subsets \( S_1, \ldots, S_q \) and \( T_1, \ldots, T_p \) as above, there is the derived matrix

\[
R_{T_1} \cdots R_{T_p} D_{S_q} \cdots D_{S_1} E.
\]

In particular, step matrices are derived matrices under the trivial action with \( S_i = T_j = \emptyset \) for all \( i, j \).

Let \( a = A_1 | A_2 | \cdots | A_q \) and \( b = B_1 | B_2 | \cdots | B_l \) be partitions of \( n \). Then \( a \times b \) is a \((p, q)\)-complementary pair (CP) if there is a \( q \times p \) derived matrix \( M = (m_{ij}) \) such that \( A_i = \{m_{ij} \neq 0 \mid 1 \leq i \leq q \} \) and \( B_i = \{m_{ij} \neq 0 \mid 1 \leq j \leq p \} \). Thus \((p, q)\)-CPs, which are in one-to-one correspondence with derived matrices, identify a particular set of product cells in \( P_n \times P_n \).

**Definition 1.** Define

\[
\Delta_p : C_0(P_1) \rightarrow C_0(P_1) \otimes C_0(P_1)
\]

by \( \Delta_p(1) = 1 \otimes 1 \). Inductively, having defined

\[
\Delta_p : C_n(P_k) \rightarrow C_n(P_k) \otimes C_n(P_k)
\]

for all \( k \leq n \), define \( \Delta_p \) on \( n + 1 \in C_n(P_{n+1}) \) by

\[
\Delta_p(n + 1) = \sum_{\text{\((p, q)\)-CPs} \ a \times b} \pm a \otimes b
\]

and extend \( \Delta_p \) to all of \( C_n(P_{n+1}) \) multiplicatively, i.e., define \( \Delta_p \) on a generator \( u_1 | \cdots | u_r \in C_{n+r+1}(P_{n+1}) \) by

\[
\Delta_p(u_1 | \cdots | u_r) = \Delta_p(u_1) \cdots \Delta_p(u_r).
\]

Recall that faces of \( P_n \) in codimension \( k \) are indexed by planar-rooted trees with \( n + 1 \) leaves and \( k + 1 \) levels (PLTs).

**Example 1.** In terms of PLTs, the diagonal \( \Delta_p \) on the top cell of \( P_3 \) (up to sign) is given by

\[
\Delta_p(\Psi) = \Psi \otimes \Psi + \Psi \otimes \Psi
\]

\[
+ \Psi \otimes \Psi + \Psi \otimes \Psi
\]

\[
+ \Psi \otimes \Psi + \Psi \otimes \Psi.
\]

The diagonal \( \Delta_p \) descends to a diagonal \( \Delta_K \) on \( C_n(K) \) via Tonks’ cellular projection \( \vartheta_0 : P_n \rightarrow K_{n+1} \), which forgets levels (see [12]). Faces of \( P_n \) indexed by PLTs with multiple nodes in the same level degenerate under \( \vartheta_0 \), and corresponding generators span the kernel of the induced map \( \vartheta_0 : C_*(P_n) \rightarrow C_*(K_{n+1}) \). The diagonal \( \Delta_K \) is given by

\[
\Delta_K \vartheta_0 = (\vartheta_0 \otimes \vartheta_0) \Delta_p.
\]

**Example 2.** When \( n = 3 \), the components \( 132 \otimes 132 \) and \( 312 \otimes 132 \) of \( \Delta_p(3) \) degenerate under \( \vartheta \) because the tree corresponding to \( 132 \) has two vertices in level 1; equivalently, \( \dim(132) = 1 \) whereas \( \dim \vartheta(132) = 0 \). Therefore (up to sign) the diagonal on the top cell of \( K_4 \) is given by

\[
\Delta_K(\Psi) = \Psi \otimes \Psi + \Psi \otimes \Psi
\]

\[
+ \Psi \otimes \Psi + \Psi \otimes \Psi
\]

\[
+ \Psi \otimes \Psi + \Psi \otimes \Psi.
\]
5. Δ-derivations and ΔX-compatible families

Let \( \{X_n\}_{n \geq 0} \) be a family of contractible polytopes such that \( \dim X_n = n \), let \( X = \bigcup_{n \geq 0} X_n \) and assume that the cellular chains \( C_\ast (X) \) are equipped with a diagonal approximation

\[
\Delta_X : C_\ast (X) \to C_\ast (X) \otimes C_\ast (X).
\]

In this section we introduce the general notion of a \( \Delta \)-derivation homotopy with respect to a \( \Delta_X \)-compatible family of maps. When \( X_n \) is the \( n \)-simplex \( s_n \), our definition agrees with the notion of a high derivation defined by Kadeishvili in [4]. When \( X_n \) is the \( n \)-dimensional permutahedron \( P_{n+1} \), the notion of a \( \Delta \)-derivation with respect to a \( \Delta_p \)-compatible family is encoded in the construction of the biderivative given by S. Saneblidze and the second author (see [10, 11, 13]).

For each \( n \), let \( n_k \) be the number of \( k \)-faces of \( X_n \) and choose a system of generators \( \{x_i^k\}_{0 \leq k \leq n; 1 \leq i \leq n_k} \) for \( C_\ast (X_n) \). Let \( X_i^k \) denote the smallest subcomplex of \( X_n \) containing the \( k \)-face associated with \( x_i^k \). Given DGAs \((A, \mu, d_A)\) and \((B, \mu_B, d_B)\), let \( \Theta : C_\ast (X_n) \to \text{Hom} (A, B) \) be a map of degree zero and let \( \Theta^k = \Theta |_{C_\ast (x_i^k)} \).

**Definition 2.** The family of maps

\[
\tilde{\Theta}_n = \{ \Theta (x_i^k) \}_{0 \leq k \leq n; 1 \leq i \leq n_k}
\]

is \( \Delta_X \)-compatible if each \( \Theta^k \) is a chain map and the following diagram commutes:

\[
\begin{align*}
C_k (X_i^k) & \xrightarrow{\Delta \Theta} \sum_{p+q=k} C_p (X_i^k) \otimes C_q (X_i^k) \\
\Theta^k & \downarrow \quad \downarrow \Theta^k \otimes \Theta^k \\
\text{Hom}^k (A, B) & \xrightarrow{\mu_A} \sum_{p+q=k} \text{Hom}^p (A, B) \otimes \text{Hom}^q (A, B) \\
& \quad \downarrow \approx \\
\text{Hom}^k (A \otimes A, B) & \xleftarrow{\mu_B} \text{Hom}^k (A \otimes A, B \otimes B).
\end{align*}
\]

Let \( \tilde{\Theta}_n \) be a \( \Delta_X \)-compatible family of maps. The map \( T = \Theta (x_i^k) : A \to B \) associated with the top dimensional cell of \( X \) is a \( \Delta \)-derivation with respect to \( \tilde{\Theta}_n \) if the diagram above commutes when \( k = n \). If in addition, \( \Theta \) is a chain map, then \( T \) is a \( \Delta \)-derivation homotopy with respect to \( \tilde{\Theta}_n \). There is the dual notion of a \( \Delta \)-coderivation homotopy with respect to a \( \Delta_X \)-compatible family.

Let \( \sigma_{n,2} : (A \otimes \cdots \otimes A) \otimes \cdots \otimes A \) be the canonical permutation of tensor factors. Then for example,

\[
\sigma_{3,2} (a_1 a_2 a_3 \otimes b_1 b_2 b_3) = (-1)^{|a_1||a_2|+|b_1||a_3|+|b_2||a_3|} a_1 b_1 \otimes a_2 b_2 \otimes a_3 b_3.
\]

**Example 3.** Set \( X_n = K_{n+2} \) and let \( \Delta_K \) be the S-U diagonal on \( C_\ast (K) \). Given a DGA \((A, d, \mu)\), let \( d^{0i} \) denote the free linear extension of \( d \) to \( A^{0i} \). Choose an arbitrary family of DG module maps \( \{\Delta_i \in \text{Hom}^{i-2} (A, A^{0i})\}_{i \geq 2} \). For notational simplicity, identify \( \Delta_i \) with the down-rooted \( i \)-leaf corolla and other down-rooted planar trees with their corresponding compositions in \( \text{Hom} (A, A^{0i}) \). When \( n = 2 \), \( K_2 \) is a point. If \( Y \) is \( \Delta \)-derivation homotopy with respect to the empty family \( \tilde{\Theta}_2 \), then \( Y \) is a DGA map, i.e.,

\[
d^{02} Y - Y d = 0 \quad \text{and} \quad Y \mu = \mu \sigma_{2,2} \mu (Y \otimes Y).
\]

When \( n = 3, K_3 \) is an interval. If \( \Theta_3 = \{ \langle \cdot, Y \rangle \} \) is a \( \Delta_K \)-compatible family of compositions, then \( Y \) and \( Y \) are DGA maps. If \( Y \) is a \( \Delta \)-derivation homotopy with respect to \( \tilde{\Theta}_3 \), then \( Y \) is a \( \langle Y, Y \rangle \)-derivation homotopy, i.e.,

\[
d^{03} Y + Y d = Y - Y \quad \text{and} \quad Y \mu = \mu^{03} \sigma_{3,2} (Y \otimes Y + Y \otimes Y).
\]
In the case of the pentagon $K_4$, assume that $\mathfrak{g}_4 = \{ \Theta \left( x^k_i \right) \}_{k=0,1}$ is a $\Delta_K$-compatible family. Then
\[
\begin{align*}
\Psi, \quad \Psi, \quad \Psi, \quad \Psi \quad \text{and} \quad \Psi \quad \text{are DGA maps};
\Psi \quad \text{is a} \quad (\Psi, \Psi) \quad \text{-derivation homotopy;}
\Psi \quad \text{is a} \quad (\Psi, \Psi) \quad \text{-derivation homotopy;}
\Psi \quad \text{is a} \quad (\Psi, \Psi) \quad \text{-derivation homotopy;}
\Psi \quad \text{is a} \quad (\Psi, \Psi) \quad \text{-derivation homotopy;}
\Psi \quad \text{is a} \quad (\Psi, \Psi) \quad \text{-derivation homotopy.}
\end{align*}
\]

If $\Psi$ is a $\Delta$-derivation homotopy with respect to $\mathfrak{g}_4$, then
\[
d^{\Psi^4} \Psi - \Psi d = (\Psi + \Psi + \Psi) - (\Psi + \Psi) \quad \text{and}
\Psi \mu = \mu^{\Psi^4} \sigma_{A,2} (\Psi \otimes \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi)
+ \Psi \otimes \Psi - \Psi \otimes \Psi,
\]
and so on.

Note that when the initial family of maps is $\{ \Delta = \Delta_2, \Delta_n \}$, the only components of $\Delta_K$ that come into play are primitive and the $\Delta_K$-compatible family $\mathfrak{g}_n$ is simply
\[
\{ f = (\Delta \otimes 1^{\otimes n-2}) \cdots (\Delta \otimes 1) \Delta, \ g = (1^{\otimes n-2} \otimes \Delta) \cdots (1 \otimes \Delta) \Delta \}.
\]
We shall refer to a $\Delta$-derivation with respect to such an $\mathfrak{g}_n$ as an "$(f, g)$-derivation of degree $n - 2$".

**Definition 3.** Let $(A, \mu_A)$ and $(B, \mu_B)$ be graded algebras and let $f, g : A \to B$ be algebra maps. A map $h : A \to B$ of degree $k$ is an $(f, g)$-derivation of degree $k$ if
\[
h \mu_A = \mu_B (f \otimes h + h \otimes g).
\]

**6. The Hopf $A_{\infty}$-coalgebra $E \otimes \Gamma$**

The natural $A_{\infty}$-bialgebra structure on tensor factors $E \otimes \Gamma$ of $H_n(\mathbb{Z}; \mathbb{Z}_p)$ is what we shall call a "Hopf $A_{\infty}$-coalgebra". Let
\[
f^n = (\Delta \otimes 1^{\otimes n-2}) \cdots (\Delta \otimes 1) \Delta.
\]

**Definition 4.** Let $n \geq 3$. A Hopf $A_{\infty}$-coalgebra is an $R$-Hopf algebra $(A, \Delta, \mu)$ together with an operation $\Delta_n \in \text{Hom}^{n-2}(A, A^{\otimes n})$ such that $(A, \Delta, \Delta_n)$ is an $A_{\infty}$-coalgebra and $\Delta_n$ is an $(f^n, f^n)$-derivation of degree $n - 2$.

Our choice of terminology in Definition 4 is motivated by the fact that the operations $\{ \Delta, \Delta_n, \mu \}$ satisfy the Hopf relation and its analogue in degree $n - 2$, i.e.,
\[
\Delta_n \mu = \mu^{\otimes n} \sigma_{n,2} (f^n \otimes \Delta_n + \Delta_n \otimes f^n).
\]

Our main result applies the following lemma, which follows from Vandermonde's identity:

**Theorem 3 (Vandermonde's Identity).** For $r, s \geq 0$ and $0 \leq k \leq r + s$,
\[
\binom{r + s}{k} = \sum_{i=0}^{k} \binom{r}{i} \binom{s}{k - i}.
\]

**Lemma 1.** Let $R = \mathbb{N} \cup \{ 0 \}$ or $R = \mathbb{Z}_p$ with $p$ prime. For all $i \geq 0$ and all $n$-tuples $(z_1, \ldots, z_n) \in R^n$ we have
\[
\binom{z_1 + \cdots + z_n + 1}{i} = \sum_{s_1 + \cdots + s_n = i-1} \binom{z_1}{s_1} \cdots \binom{z_n}{s_n} + \sum_{t_1 + \cdots + t_n = i} \binom{z_1}{t_1} \cdots \binom{z_n}{t_n},
\]
where we reduce mod $p$ when $R = \mathbb{Z}_p$. 

Proof. A standard formula for binomial coefficients gives
\[
\binom{z_1 + \cdots + z_n + 1}{i} = \binom{z_1 + \cdots + z_n}{i-1} + \binom{z_1 + \cdots + z_n}{i}.
\]
Iteratively apply Vandermonde’s identity to the right-hand summand and obtain
\[
\binom{z_1 + \cdots + z_n}{i} = \sum_{k_1=0}^{i} \binom{z_1}{k_1} \binom{z_2}{k_1-1} \cdots \binom{z_n}{k_1-1}. \tag{6.2}
\]
Note that the sum of the lower entries in the \(n \) binomial coefficients of this last expression is \( i \) and set \( t_1 = k_{n-1}, t_2 = k_{n-1} - k_{n-2}, \ldots, t_{n-1} = k_1 - k_2 \) and \( t_n = i - k_1 \).

Then expression (6.2) can be rewritten as
\[
\binom{z_1 + \cdots + z_n}{i} = \sum_{t_1 + \cdots + t_n = i} (z_1) \cdots (z_n) t_n \tag{6.2}.
\]

Remark 1. The formula in Lemma 1 counts the number of ways \( i \) objects can be selected from a collection of \( z_1 + \cdots + z_p + 1 \) objects of \( n + 1 \) different colors, one of which is uniquely colored “black” and \( z_i \) of which have the same unique color for each \( i \). The first sum on the right-hand side counts the ways to select \( i \) objects one of which is black; the second sum counts the ways to select \( i \) objects none of which are black.

Theorem 4. Let \( p \) be an odd prime. For each \( i \geq 0 \) and \( n \geq 1 \), let
\[
A_i = E(v_1, 2np^j + 1) \oplus \Gamma(w_1, 2np^{j+1} + 2),
\]
let \( \mu = (\mu_E \otimes \mu_f)\sigma_{2,2} \), and for \( j = 2, p \), let \( \Delta_j \) be defined as in (3.1) and (3.2). Then \((A_i, \Delta_2, \Delta_p, \mu)\) is a Hopf \( A_\infty \)-coalgebra over \( Z_p \).

Proof. For each \( n \geq 1 \) and \( i \geq 0 \), \((A_i, \Delta_2, \mu)\) is a Hopf algebra via \( \mu = (\mu_E \otimes \mu_f)\sigma_{2,2} \), where
\[
\mu_E(v_1 \otimes v_1) = 0 \quad \text{and} \quad \mu_f((\gamma_i(w_i) \otimes \gamma_j(w_i))) = \binom{i+j}{i} \gamma_{i+j}(w_i),
\]
and \((A_i, \Delta_2, \Delta_p)\) is an \( A_\infty \)-coalgebra by Theorem 2. Furthermore, both sides of relation (6.1) vanish on tensor products involving \( v_i \), and it is sufficient to evaluate this relation on \( \gamma_i \otimes \gamma_j = \gamma_i(w_i) \otimes \gamma_j(w_i) \). First,
\[
\Delta_p \mu(\gamma_i \otimes \gamma_j) = \binom{i+j}{i} \Delta_p(\gamma_{i+j}) = \sum_{z_1 + \cdots + z_p = i+j-1} \binom{i+j}{i} v_1 \gamma_{z_1} \otimes \cdots \otimes v_1 \gamma_{z_p} = \sum_{z_1 + \cdots + z_p = i+j} \binom{z_1 + \cdots + z_p + 1}{i} u, \tag{6.3}
\]
where \( u = v_1 \gamma_{z_1} \otimes \cdots \otimes v_1 \gamma_{z_p} \), and second,
\[
\mu^{op}_{2,2} (f_p \otimes \Delta_p + \Delta_p \otimes f_p)(\gamma_i \otimes \gamma_j) = \sum_{m_1 + \cdots + m_p = i+j-1} \binom{m_1 + t_1}{t_1} \cdots \binom{m_p + t_p}{t_p} v_i \gamma_{m_1+t_1} \otimes \cdots \otimes v_i \gamma_{m_p+t_p} \]
\[
= \sum_{z_1 + \cdots + z_p = i+j} \left[ \sum_{s_1 + \cdots + s_p = i} \binom{z_1}{s_1} \cdots \binom{z_p}{s_p} + \sum_{t_1 + \cdots + t_p = i} \binom{z_1}{t_1} \cdots \binom{z_p}{t_p} \right] u. \tag{6.4}
\]
But expressions (6.3) and (6.4) are equal modulo \( p \) by Lemma 1. □

In view of Eilenberg and Mac Lane’s decomposition of \( H_*(Z, n; Z_p) \) discussed in Section 3, the submodule
\[
A_i = E(v_1, 2np^j + 1) \oplus \Gamma(w_1, 2np^{j+1} + 2) \subset H_*(Z, n; Z_p)
\]
is a naturally occurring Hopf \( A_\infty \)-coalgebra for each \( i \geq 0 \). Furthermore, when \( n = 3 \), the Hopf \( A_\infty \)-coalgebra structures on the \( A_i \)’s induce a global \( A_\infty \)-bialgebra structure on \( H_*(Z, 3; Z_p) \) as a formal (as yet undefined) tensor product of \( A_\infty \)-bialgebras. The relationship between this global structure and the underlying topology of \( K(Z, 3) \) is an interesting open question.
Acknowledgements

The work of the first author was partially supported by project “EHU09/04” and by the PAICYT research project FQM-296. The research of the second author was funded in part by a Millersville University faculty research grant.

References