
Edge Tessellations and Stamp Folding Puzzles

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Which polygons generate a tiling of the plane when reflected in their edges? The complete answer was discovered by Millersville University students Andrew Hall, Joshua York, and the first author in the spring of 2009, and we present it here as a theorem.

THEOREM 1. *A polygon generating a tiling of the plane when reflected in its edges is one of the following eight types: a rectangle; an equilateral, 60-right, isosceles right, or 120-isosceles triangle; a 120-rhombus; a 60-90-120 kite; or a regular hexagon.*

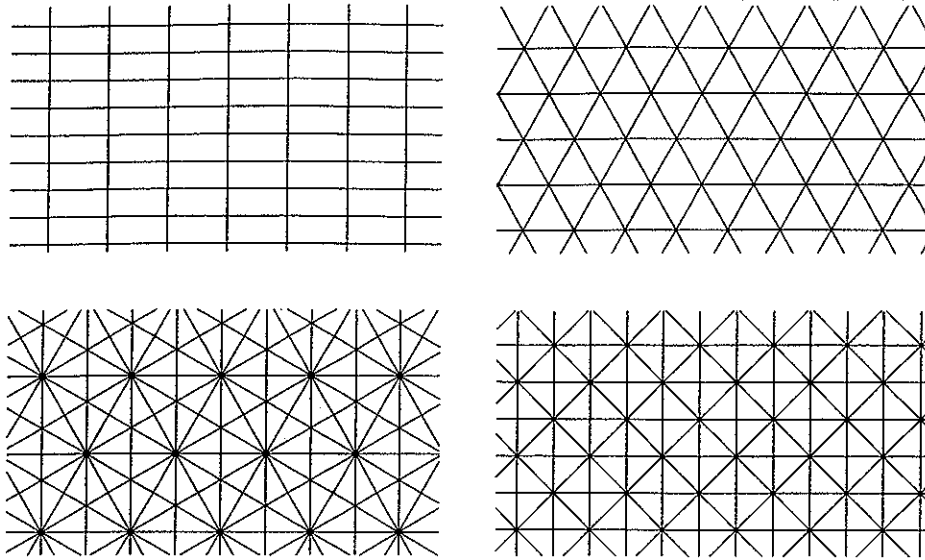


Figure 1 Edge tessellations generated by non-obtuse polygons

A *tessellation* (or *tiling*) of the plane is a collection of plane figures that fills the plane with no overlaps and no gaps. An *edge tessellation* is generated by reflecting a polygon in its edges. The eight edge tessellations, which are pictured in FIGURES 1 and 2, are the most symmetric examples of *Laves tilings*. A discussion of Laves tilings and a complete list of them are given by Grünbaum and Shephard [4].

Edge tessellations provide the setting for *stamp folding puzzles*, which are paper folding problems constrained to the perforations on a sheet of postage stamps. The sheet must embed in an edge tessellation, may have any shape, and may be bounded or unbounded, with bounding edges along perforations as in FIGURE 3. The object of a stamp folding puzzle is to create some specified configuration by folding the sheet along its perforations without creasing the stamps. Tucks, which slip one subpacket of

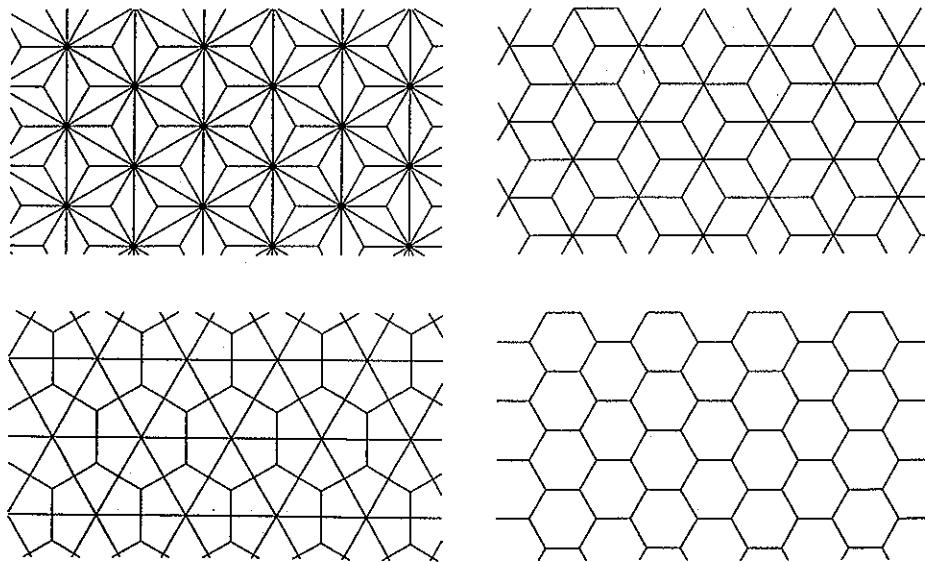


Figure 2 Edge tessellations generated by obtuse polygons

folded stamps between the leaves of another, are allowed. Indeed, a tuck is required to solve the following delightful problem posed by G. Frederickson in his book *Piano-Hinged Dissections: Time to Fold!* [3, p. 144]:

Consider the block of sixteen isosceles right triangular stamps pictured in FIGURE 3. Fold the block into a packet sixteen-deep so that the stamps are arranged in the order 4, 1, 16, 6, 5, 15, 14, 8, 7, 13, 11, 12, 2, 3, 9, 10.

A sheet of postage stamps is suitable for stamp folding puzzles if the stamps are configured in such a way that the sheet folds neatly into a packet of single stamps. Such a packet unfolds into an edge tessellation in which the perforations form lines of symmetry. Frederickson poses the following conjecture [3, p. 143]:

Although triangular stamps have come in a variety of different triangular shapes, only three shapes seem suitable for [stamp] folding puzzles: equilateral, isosceles right, and 60° -right triangles.

Theorem 1 confirms Fredickson's conjecture; indeed, the four suitable edge tessellations are pictured in FIGURE 1. We invite the reader to reproduce and fold each of them into a packet of single stamps. Our folding algorithms appear at the end of this article.

The question posed at the outset of this article and answered by Theorem 1, was motivated by the "unfolding technique" applied by A. Baxter and the second author to find, classify and count classes of periodic orbits of a billiard ball in motion on an equilateral triangular billiard table (see [2] for details). Periodic orbits on polygonal billiard tables of the eight polygonal types in Theorem 1 unfold as straight line segments in an edge tessellation. During an REU in 2001, Andrew Baxter and students Ethan McCarthy and Jonathan Eskreis-Winkler applied this unfolding technique to find, classify, and count classes of periodic orbits on square, rectangular, and isosceles right triangular billiard tables (see [1]). Presumably, this technique also applies on billiard tables of the five remaining polygonal types.

Edge tessellations are *wallpaper patterns*, which are tessellations of the plane with translational symmetries of minimal length in two independent directions (the group of symmetries is discrete). A point C in a wallpaper pattern is an n -center if the group of rotational symmetries centered at C is generated by a rotation of minimal positive rotation angle $\phi_n = 360^\circ/n$.

The students' original proof of Theorem 1 applies the powerful *Crystallographic Restriction Theorem*, which tightly constrains the order of a group of rotational symmetries: *If C is an n -center of a wallpaper pattern, then $n \in \{2, 3, 4, 6\}$* (for a proof,

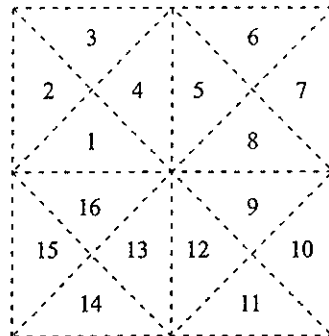


Figure 3 A block of sixteen isosceles right triangular stamps

see [5] for example). The proof of Theorem 1 presented here is independent of Crystallographic Restriction and more geometrically revealing.

Proof. We begin the proof of Theorem 1 by constructing a set S containing the measures of the interior angles of a generating polygon G . Let V be a vertex of G , and let θ be the measure of the interior angle at V ; then $\theta < 180^\circ$. Let G' be the image of G when reflected in an edge of G containing V . Then the interior angle of G' at V has measure θ , and inductively, the interior angle at V of every copy of G with vertex V has measure θ (see FIGURE 4). Since successively reflecting in the edges of G that meet at V is a rotational symmetry of angle 2θ , the vertex V is an n -center for some n . If G' is the rotational image of G , then $\phi_n = \theta$; otherwise $\phi_n = 2\theta$. In either case, $n\theta = 360^\circ$ for some $n \in \mathbb{N}$, and it follows that every interior angle of G lies in the set

$$\begin{aligned} S &= \{x \leq 120^\circ \mid nx = 360^\circ, n \in \mathbb{N}\} \\ &= \{120^\circ, 90^\circ, 72^\circ, 60^\circ, 51\frac{3}{7}^\circ, 45^\circ, 40^\circ, 36^\circ, \dots, 18^\circ, \dots\}. \end{aligned}$$

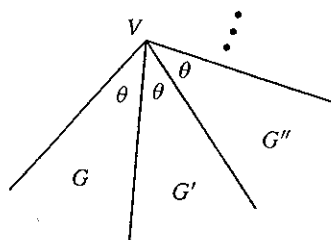


Figure 4 Congruent interior angles at vertex V shared by G and images G' and G''

Now suppose that $\theta = 120^\circ$; then three copies of G share the vertex V . Let e and e' be the edges of G that meet at V , and labeled so that the angle from e to e' measures 120° (see FIGURE 5). Let e' and e'' be their respective images under a 120° rotation. Then e'' lies on the bisector of $\angle V$ and is the reflection of e' in e . By a similar argument, if an odd number of copies of G share vertex V , the bisector of $\angle V$ is a line of symmetry.

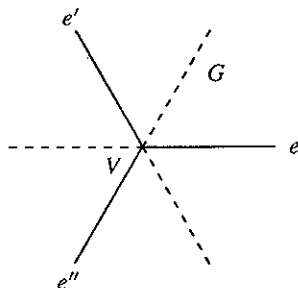


Figure 5 A line of symmetry bisects a 120° interior angle

Let g be the number of edges of G . Then the interior angle sum $180^\circ(g - 2) \leq 120^\circ g$ implies $g \leq 6$. If G is a hexagon, it is equiangular since its interior angles measure at most 120° and its interior angle sum is $720^\circ = 6(120^\circ)$. But G is symmetric with respect to each of its interior angle bisectors by the remark above. Therefore G is a *regular hexagon*.

We claim that G is not a pentagon. On the contrary, suppose G is a pentagon. Then some interior angle measures 120° since the interior angle sum of $540^\circ > 5(90^\circ)$. Choose an interior angle of 120° and label the vertex at this interior angle V . Then G is symmetric with respect to the angle bisector at V and the other interior angles of G pair off congruently—two with measure x , two with measure y . Note that the interior angles in one of these pairs are adjacent (see FIGURE 6). If $x = y$, then $x = 105^\circ \notin S$; hence $x \neq y$. If $x < y$, then $y > 105^\circ$; hence $y = 120^\circ$ since $y \in S$. But if $y = 120^\circ$, lines of symmetry bisect three interior angles of G , in which case $x = y$, which is a contradiction.

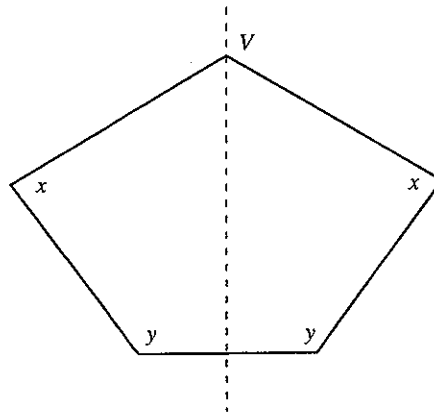


Figure 6 A pentagon G with an interior angle of 120° at V

Now if V is a vertex of G , let $m\angle V$ denote the measure of the interior angle at V . Suppose G is a quadrilateral. If G has an interior angle of 120° , label the vertices A, B, C, D in succession with $m\angle A = 120^\circ$. Then the bisector s of $\angle A$ is a line of symmetry, C is on s , and $\angle B \cong \angle D$ (see FIGURE 7). Let $2x = m\angle C$ and $y = m\angle B$, and note that $m\angle BAC = 60^\circ$. Then $x \leq 60^\circ \leq 120^\circ - x = y$. Hence the only solutions of $x + y = 120^\circ$ with $x \leq y$ and $(x, y) \in S \times S$ are $\{(30^\circ, 90^\circ), (60^\circ, 60^\circ)\}$. Therefore G is either a *120-rhombus* or a *60-90-120 kite*. On the other hand, if the interior angles of G measure at most 90° , then G is equiangular since its interior angle sum is 360° , and G is a *rectangle*.

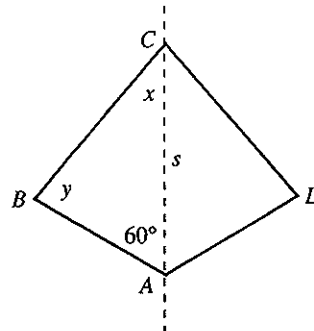


Figure 7 A quadrilateral G with an obtuse interior angle

Finally, suppose G is a triangle. If G has an interior angle of 120° , then G is a 120-isosceles triangle by symmetry. Otherwise, let $G = \triangle ABC$; let $x = m\angle A$, $y = m\angle B$, and $z = m\angle C$.

If G is a right triangle with $z = 90^\circ$ and $x \leq y$, then $x, y \in S$ implies $18^\circ \leq x \leq y \leq 72^\circ$. Hence the only solutions of $x + y = 90^\circ$ with $x \leq y$ and $(x, y) \in S \times S$ are $\{(18^\circ, 72^\circ), (30^\circ, 60^\circ), (45^\circ, 45^\circ)\}$. Furthermore, if $y = 72^\circ$, five copies of G share vertex B and the bisector of $\angle B$ is a line of symmetry, in which case $x = z = 90^\circ$, which is a contradiction. Therefore G is either a 60-right or an isosceles right triangle.

If G is an acute triangle with $x \leq y \leq z \leq 72^\circ$, then $x = 180^\circ - (y + z) \geq 180^\circ - 2(72^\circ) = 36^\circ$; on the other hand, $x = 180^\circ - (y + z) \leq 180^\circ - 2(60^\circ) = 60^\circ$. But $36^\circ \leq x \leq 60^\circ$ implies $120^\circ \leq y + z \leq 144^\circ$. Thus if $y \leq z$ and $(y, z) \in S \times S$, then $(y, z) \in \{(60^\circ, 60^\circ), (60^\circ, 72^\circ), (72^\circ, 72^\circ)\}$ so that the only solutions of $x + y + z = 180^\circ$ with $x \leq y \leq z$ and $(x, y, z) \in S \times S \times S$ are $\{(36^\circ, 72^\circ, 72^\circ), (60^\circ, 60^\circ, 60^\circ)\}$. But interior angles of 72° are bisected by lines of symmetry, so the solution $(36^\circ, 72^\circ, 72^\circ)$ is extraneous and G is an equilateral triangle.

This completes the proof of Theorem 1. ■

We remark that edge tessellations represent 3 of the 17 symmetry types of wallpaper patterns. Using the labeling defined in [5], general (non-square) rectangles generate patterns of type pmm ; isosceles right triangles and squares generate patterns of type $p4m$; and the other six polygons in Theorem 1 generate patterns of type $p6m$.

Here are some explicit algorithms for folding the sheets of stamps in FIGURE 1 into packets of single stamps. Let T be generated by one of the four non-obtuse polygons identified in Theorem 1. Choose an infinite strip S of minimal width bounded by parallel perforation lines l and m , and "accordion-fold" T onto S , i.e., fold along l then along m so that S has four leaves configured as a "w," then fold again along l and again along m so that S has eight "zig-zag" leaves, and continue in this manner indefinitely.

Choose a stamp P in S . If P is a rectangle, accordion-fold S onto P . If P is a right triangle, P together with some subset of its images tessellate a rectangle R of minimal area contained in S , so accordion-fold S onto R , then fold R onto P . If P is an equilateral triangle, two of its edges lie in the interior of S . Label these edges a and b , then its third edge c is contained in the boundary of S . Let S_1 and S_2 be the subsets of $S - P$ bounded by l, m , and a , and by l, m , and b , respectively. Fold S_1 along a , then along c , then along b , and continue in this manner indefinitely to form an infinite "spiral"; similarly, fold S_2 along b , then along c , then along a , and continue indefinitely.

On the other hand, if T is generated by an obtuse polygon G , it has a 3-center C shared by three copies of G . Since the interior angle of G at C is bisected by a line of symmetry l , which contains an edge of some copy of G , folding along l creases the stamp G . Thus T is not suitable for stamp folding puzzles, and we have established Frederickson's conjecture:

THEOREM 2. *The edge tessellations suitable for stamp folding puzzles are generated by the four non-obtuse polygons indicated in Theorem 1.*

To summarize, we have proved that a polygon generating an edge tessellation is one of the following eight types: a rectangle; an equilateral, 60-right, isosceles right, or 120-isosceles triangle; a 120-rhombus; a 60-90-120 kite; or a regular hexagon. Of these, the four non-obtuse polygons generate tessellations suitable for stamp folding puzzles. Our proof of suitability, which establishes Frederickson's Conjecture, exhibits explicit algorithms for folding each suitable edge tessellation into a packet of single stamps.

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REFERENCES

1. A. Baxter, J. Eskreis-Winkler, and E. McCarthy, Periodic Billiard Paths in Edge-Tessellating Polygons, unpublished manuscript.
2. A. Baxter and R. Umble, Periodic Orbits for Billiards on an Equilateral Triangle, *Amer. Math. Monthly*, **115** (6) (2008) 479–491.
3. G. N. Frederickson, *Piano-Hinged Dissections: Time to Fold!* A. K. Peters, Ltd., Wellesley, MA, 2006.
4. B. Grünbaum and G. Shephard, *Tilings and Patterns*, W. H. Freeman and Company, New York, 1986.
5. G. E. Martin, *Transformation Geometry: An Introduction to Symmetry*, Springer-Verlag, New York, 1982.

Summary An edge tessellation is a tiling of the plane generated by reflecting a polygon in its edges. In this article we prove that a polygon generating an edge tessellation is one of the following eight types: a rectangle; an equilateral, 60°-right, isosceles right, or 120°-isosceles triangle; a 120°-rhombus; a 60°-90°-120° kite; or a regular hexagon. A stamp folding puzzle is a paper folding problem constrained to the perforations on a sheet of postage stamps. Such sheets necessarily embed in an edge tessellation. On page 143 of his book *Piano-Hinged Dissections: Time to Fold!*, G. Frederickson poses the following conjecture: “Although triangular stamps have come in a variety of different triangular shapes, only three shapes seem suitable for [stamp] folding puzzles: equilateral, isosceles right triangles, and 60°-right triangles.” We prove that the four non-obtuse polygons mentioned above generate edge tessellations suitable for stamp folding puzzles. Our proof of suitability, which establishes Frederickson’s Conjecture, exhibits explicit algorithms for folding each suitable edge tessellation into a packet of single stamps.
