
Periodic Orbits for Billiards on an Equilateral Triangle

Andrew M. Baxter and Ronald Umble

1. INTRODUCTION. The trajectory of a billiard ball in motion on a frictionless billiards table is completely determined by its initial position, direction, and speed. When the ball strikes a bumper, we assume that the angle of incidence equals the angle of reflection. Once released, the ball continues indefinitely along its trajectory with constant speed unless it strikes a vertex, at which point it stops. If the ball returns to its initial position with its initial velocity direction, it retraces its trajectory and continues to do so repeatedly; we call such trajectories *periodic*. Nonperiodic trajectories are either *infinite* or *singular*; in the later case the trajectory terminates at a vertex.

More precisely, think of a billiards table as a plane region Ω bounded by a polygon π . A *nonsingular trajectory on π* is a piecewise linear constant speed curve $\alpha : \mathbb{R} \rightarrow \Omega$, where $\alpha(t)$ is the position of the ball at time t . An *orbit* is the restriction of some nonsingular trajectory to a closed interval; this is distinct from the notion of “orbit” in discrete dynamical systems.

A nonsingular trajectory α is *periodic* if $\alpha(a + t) = \alpha(b + t)$ for some $a < b$ and all $t \in \mathbb{R}$; its restriction to $[a, b]$ is a *periodic orbit*. A periodic orbit retraces the same path exactly $n \geq 1$ times. If $n = 1$, the orbit is *primitive*; otherwise it is an *n-fold iterate*. If α is primitive, α^n denotes its *n-fold iterate*. The *period* of a periodic orbit is the number of times the ball strikes a bumper as it travels along its trajectory. If α is primitive of period k , then α^n has period kn .

In this article we give a complete solution to the following billiards problem: *Find, classify, and count the classes of periodic orbits of a given period on an equilateral triangle*. While periodic orbits are known to exist on all nonobtuse and certain classes of obtuse triangles [5], [8], [11], [14], existence in general remains a long-standing open problem. The first examples of periodic orbits were discovered by Fagnano in 1745. Interestingly, his orbit of period 3 on an acute triangle, known as the “Fagnano orbit,” was not found as the solution of a billiards problem, but rather as the triangle of least perimeter inscribed in a given acute triangle. This problem, known as “Fagnano’s problem,” is solved by the orthic triangle, whose vertices are the feet of the altitudes of the given triangle (see Figure 1). The orthic triangle is a periodic trajectory since its angles are bisected by the altitudes of the triangle in which it is inscribed; the proof given by Coxeter and Greitzer [1] uses exactly the “unfolding” technique we apply below. Coxeter credits this technique to H. A. Schwarz and mentions that Frank and

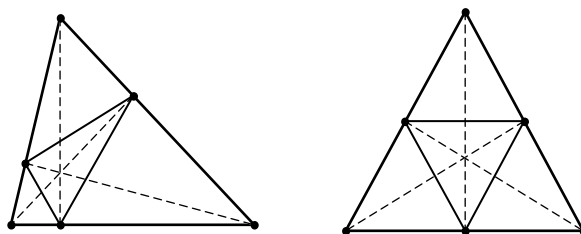


Figure 1. Fagnano’s period 3 orbit.

F. V. Morley [9] extended Schwarz's treatment on triangles to odd-sided polygons. For a discussion of some interesting properties of the Fagnano orbit on any acute triangle, see [4].

Much later, in 1986, Masur [8] proved that every rational polygon (one whose interior angles are rational multiples of π) admits infinitely many periodic orbits with distinct periods, but he neither constructed nor classified them. A year later Katok [6] proved that the number of periodic orbits of a given period grows subexponentially. Existence results on various polygons were compiled by Tabachnikov [13] in 1995.

This article is organized as follows: In Section 2 we introduce an equivalence relation on the set of all periodic orbits on an equilateral triangle and prove that every orbit with odd period is an odd iterate of Fagnano's orbit. In Section 3 we use techniques from analytic geometry to identify and classify all periodic orbits. The paper concludes with Section 4, in which we derive two counting formulas: First, we establish a bijection between classes of orbits with period $2n$ and partitions of n with 2 or 3 as parts and use it to show that there are $\mathcal{O}(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$ classes of orbits with period $2n$ (counting iterates). Second, we show that there are $\mathcal{P}(n) = \sum_{d|n} \mu(d) \mathcal{O}(n/d)$ classes of primitive orbits with period $2n$, where μ denotes the Möbius function.

2. ORBITS AND TESSELLATIONS. Consider an equilateral triangle $\triangle ABC$. We begin with some key observations.

Proposition 1. *Every nonsingular trajectory strikes some side of $\triangle ABC$ with an angle of incidence in the range $30^\circ \leq \theta \leq 60^\circ$.*

Proof. Given a nonsingular trajectory α , choose a point P_1 at which α strikes $\triangle ABC$ with angle of incidence θ_1 . If θ_1 lies in the desired range, set $\theta = \theta_1$. Otherwise, let α_1 be the segment of α that connects P_1 to the next strike point P_2 and label the vertices of $\triangle ABC$ so that P_1 is on side \overline{AC} and P_2 is on side \overline{BC} (see Figure 2). If $0^\circ < \theta_1 < 30^\circ$, then $\theta_2 = m\angle P_1P_2B = \theta_1 + 60^\circ$ so that $60^\circ < \theta_2 < 90^\circ$. Let α_2 be the segment of α that connects P_2 to the next strike point P_3 . Then the angle of incidence at P_3 satisfies $30^\circ < \theta_3 < 60^\circ$; set $\theta = \theta_3$. If $60^\circ < \theta_1 \leq 90^\circ$ and θ_1 is an interior angle of $\triangle P_1P_2C$, then the angle of incidence at P_2 is $\theta_2 = m\angle P_1P_2C = 120^\circ - \theta_1$ and satisfies $30^\circ \leq \theta_2 < 60^\circ$; set $\theta = \theta_2$. But if $60^\circ < \theta_1 \leq 90^\circ$ and θ_1 is an exterior angle of $\triangle P_1P_2C$, then the angle of incidence at P_2 is $\theta_2 = m\angle P_1P_2C = \theta_1 - 60^\circ$, in which case $0^\circ < \theta_2 \leq 30^\circ$. If $\theta_2 = 30^\circ$ set $\theta = \theta_2$; otherwise continue as above until $30^\circ < \theta_4 < 60^\circ$ and set $\theta = \theta_4$. ■

Let α be an orbit of period n on $\triangle ABC$ oriented so that \overline{BC} is horizontal. Since Proposition 1 applies equally well to periodic orbits, choose a point P at which α

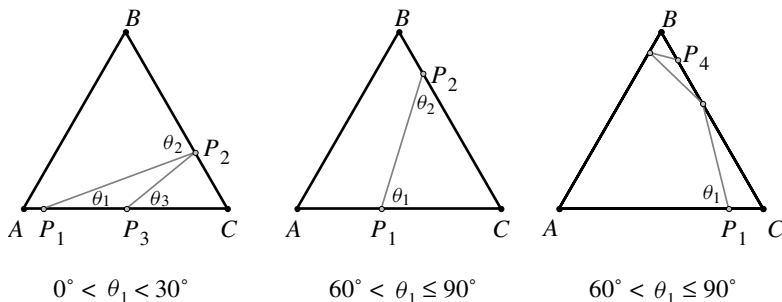


Figure 2. Incidence angles in the range $30^\circ \leq \theta \leq 60^\circ$.

strikes $\triangle ABC$ with angle of incidence in the range $30^\circ \leq \theta \leq 60^\circ$. If necessary, relabel the vertices of $\triangle ABC$, change initial points, and reverse the parameter so that side \overline{BC} contains P , α begins and ends at P , and the components of α' as the ball departs from P are positive. Let \mathcal{T} be a regular tessellation of the plane by equilateral triangles, each congruent to $\triangle ABC$, and positioned so that one of its families of parallel edges is horizontal. Embed $\triangle ABC$ in \mathcal{T} so that its base \overline{BC} is collinear with a horizontal edge of \mathcal{T} . Let $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the directed segments of α , labelled sequentially; then α_1 begins at P and terminates at P_1 on side s_1 of $\triangle ABC$ with angle of incidence θ_1 . Let σ_1 be the reflection in the edge of \mathcal{T} containing s_1 . Then α_1 and $\sigma_1(\alpha_2)$ are collinear segments and $\sigma_1(\alpha)$ is a periodic orbit on $\sigma_1(\triangle ABC)$, which is the basic triangle of \mathcal{T} sharing side s_1 with $\triangle ABC$. Follow $\sigma_1(\alpha_2)$ from P_1 until it strikes side s_2 of $\sigma_1(\triangle ABC)$ at P_2 with incidence angle θ_2 . Let σ_2 be the reflection in the edge of \mathcal{T} containing s_2 ; then $\alpha_1, \sigma_1(\alpha_2)$ and $(\sigma_2\sigma_1)(\alpha_3)$ are collinear segments and $(\sigma_2\sigma_1)(\alpha)$ is a periodic orbit on $(\sigma_2\sigma_1)(\triangle ABC)$. Continuing in this manner for $n - 1$ steps, let θ_n be the angle of incidence at $Q = (\sigma_{n-1}\sigma_{n-2} \cdots \sigma_1)(P)$. Then $\theta_1, \theta_2, \dots, \theta_n$ is a sequence of incidence angles with $30^\circ \leq \theta_n \leq 60^\circ$, and $\alpha_1, \sigma_1(\alpha_2), \dots, (\sigma_{n-1}\sigma_{n-2} \cdots \sigma_1)(\alpha_n)$ is a sequence of collinear segments whose union is the directed segment from P to Q . Using the notation in [7], let \underline{PQ} denote the directed segment from P to Q . Then \underline{PQ} has the same length as α and enters and exits the triangle $(\sigma_i \cdots \sigma_1)(\triangle ABC)$ with angles of incidence θ_i and θ_{i+1} . We refer to \underline{PQ} as an *unfolding* of α and to θ_n as its *representation angle*. See Figure 4 for examples of unfoldings.

Proposition 2. *A periodic orbit strikes the sides of $\triangle ABC$ with at most three incidence angles, exactly one of which lies in the range $30^\circ \leq \theta \leq 60^\circ$. In fact, exactly one of the following holds:*

- (1) *All incidence angles measure 60° .*
- (2) *There are exactly two distinct incidence angles measuring 30° and 90° .*
- (3) *There are exactly three distinct incidence angles $\phi, \theta,$ and ψ such that $0^\circ < \phi < 30^\circ < \theta < 60^\circ < \psi < 90^\circ$.*

Proof. Let α be a periodic orbit and let \underline{PQ} be an unfolding. By construction, \underline{PQ} cuts each horizontal edge of \mathcal{T} with angle of incidence in the range $30^\circ \leq \theta \leq 60^\circ$. Consequently, \underline{PQ} cuts a left-leaning edge of \mathcal{T} with angle of incidence $\phi = 120^\circ - \theta$ and cuts a right-leaning edge of \mathcal{T} with angle of incidence $\psi = 60^\circ - \theta$ (see Figure 3). In particular, if $\theta = 60^\circ$, \underline{PQ} cuts only left-leaning and horizontal edges, and all incidence angles are equal. In this case, α is either the Fagnano orbit, a primitive orbit of period 6, or some iterate of these. If $\theta = 30^\circ$, then $\phi = 90^\circ$ and $\psi = 30^\circ$, and α is either primitive of period 4 or some iterate thereof (see Figure 4). When $30^\circ < \theta < 60^\circ$, clearly $0^\circ < \phi < 30^\circ$ and $60^\circ < \psi < 90^\circ$. ■

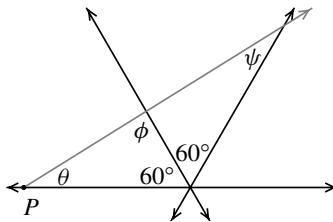


Figure 3. Incidence angles $\theta, \phi,$ and ψ .

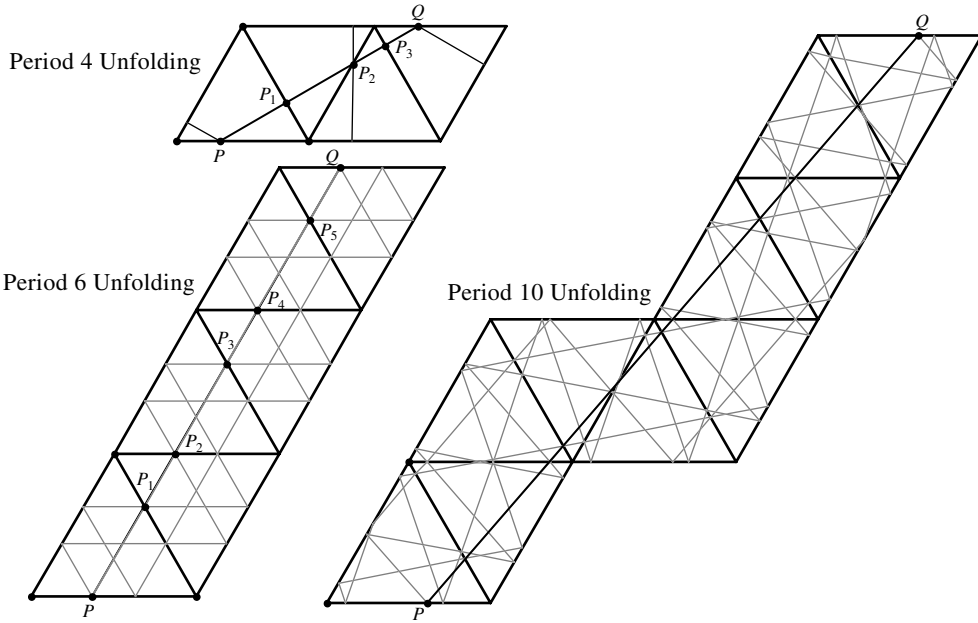


Figure 4. Unfolded orbits of period 4, 6, and 10.

Corollary 1. Any two unfoldings of a periodic orbit are parallel.

Our next result plays a pivotal role in the classification of orbits.

Theorem 1. If an unfolding of a periodic orbit α terminates on a horizontal edge of T , then α has even period.

Proof. Let \overline{PQ} be an unfolding of α . Then both P and Q lie on horizontal edges of T , and the basic triangles of T cut by \overline{PQ} pair off and form a polygon of rhombic tiles containing \overline{PQ} (see Figure 5). As the path \overline{PQ} traverses this polygon, it enters each rhombic tile through an edge, cuts a diagonal of that tile (collinear with a left-leaning edge of T), and exits through another edge. Since each exit edge of one tile is the entrance edge of the next and the edge containing P is identified with the edge

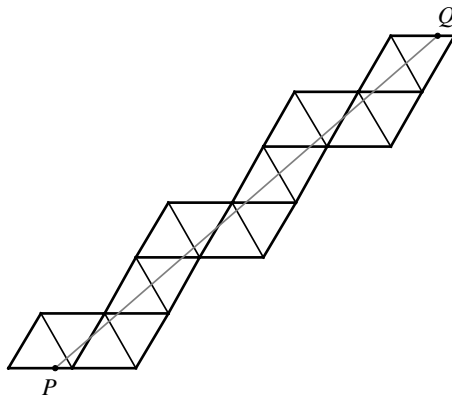


Figure 5. A typical rhombic tiling.

containing Q , the number of distinct edges of \mathcal{T} cut by \underline{PQ} is twice the number of rhombic tiles. It follows that α has even period. ■

Let γ denote the Fagnano orbit.

Theorem 2. *If α is a periodic orbit and $\alpha \neq \gamma^{2k-1}$ for all $k \geq 1$, then every unfolding of α terminates on a horizontal edge of \mathcal{T} .*

Proof. We prove the contrapositive. Suppose there is an unfolding \underline{PQ} of α that does not terminate on a horizontal edge of \mathcal{T} . Let θ be the angle of incidence at Q ; then θ is also the angle of incidence at P and $\theta \in \{30^\circ, 60^\circ\}$ by the proof of Proposition 2. But if $\theta = 30^\circ$, then α is some iterate of the period 4 orbit whose unfoldings terminate on a horizontal edge of \mathcal{T} (see Figure 4). So $\theta = 60^\circ$. But α is neither an iterate of a period 6 orbit nor an even iterate of γ since their unfoldings also terminate on a horizontal edge of \mathcal{T} (see Figure 4). It follows that $\alpha = \gamma^{2k-1}$ for some $k \geq 1$. ■

Combining the contrapositives of Theorems 1 and 2 we obtain:

Corollary 2. *If α is an orbit with odd period, then $\alpha = \gamma^{2k-1}$ for some $k \geq 1$, in which case the period is $6k - 3$.*

Let α be an orbit with even period and let \underline{PQ} be an unfolding. Let G be the group generated by all reflections in the edges of \mathcal{T} . Since the action of G on \overline{BC} generates a regular tessellation \mathcal{H} of the plane by hexagons, α terminates on some horizontal edge of \mathcal{H} . As in the definition of an unfolding, let $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ be the reflections in the lines of \mathcal{T} cut by \underline{PQ} (in order) and σ_n be the reflection in the line of \mathcal{T} containing Q . Then the composition $f = \sigma_n \sigma_{n-1} \cdots \sigma_1$ maps P to Q and maps the hexagon whose base \overline{BC} contains P to the hexagon whose base $\overline{B'C'}$ contains Q . Then n (the period of α) is even and f is either a translation by vector \overrightarrow{PQ} or a rotation of 120° or 240° . But $\overline{BC} \parallel \overline{B'C'}$ so f is a translation and the position of Q on $\overline{B'C'}$ is exactly the same as the position of P on \overline{BC} .

Periodic orbits represented by horizontal translations of an unfolding \underline{PQ} are generically distinct, but have the same period, Euclidean length, and incidence angles (up to permutation) as α . Hence it is natural to think of them as equivalent.

Definition 1. Periodic orbits α and β are *equivalent* if there exist respective unfoldings \underline{PQ} and \underline{RS} and a horizontal translation τ such that $\underline{RS} = \tau(\underline{PQ})$. The symbol $[\alpha]$ denotes the equivalence class of α . The *period* of a class $[\alpha]$ is the period of its elements; a class is *even* if and only if it has even period.

Consider an unfolding \underline{PQ} of a periodic orbit α . If $[\alpha]$ is even, let R be a point on \overline{BC} and let τ be the translation from P to R . We say that the point R is *singular* for $[\alpha]$ if $\tau(\underline{PQ})$ contains a vertex of \mathcal{T} ; then $\tau(\underline{PQ})$ is an unfolding of a periodic orbit whenever R is non-singular for $[\alpha]$. Furthermore, α strikes \overline{BC} at finitely many points and at most finitely many points on \overline{BC} are singular for $[\alpha]$. Therefore $[\alpha]$ has cardinality \mathfrak{c} (the cardinality of an interval). On the other hand, Corollary 2 tells us that an orbit of odd period is γ^{2k-1} for some $k \geq 1$. But if $k \neq \ell$, then γ^{2k-1} and $\gamma^{2\ell-1}$ have different periods and cannot be equivalent. Therefore $[\gamma^{2k-1}]$ is a singleton class for each k . We have proved:

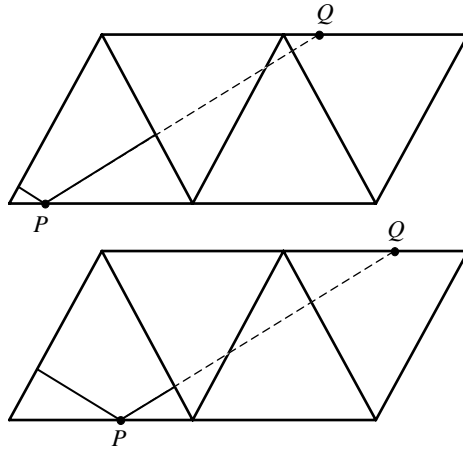


Figure 6. Unfoldings of equivalent period 4 orbits.

Proposition 3. *The cardinality of a class is determined by its parity; in fact, α has odd period if and only if $[\alpha]$ is a singleton class.*

Proposition 3 and Corollary 2 completely classify orbits with odd period. The remainder of this article considers orbits with even period. Our strategy is to represent the classes of all such orbits as lattice points in some “fundamental region,” which we now define. First note that any two unfoldings whose terminal points lie on the same horizontal edge of \mathcal{H} are equivalent. Since \mathcal{H} has countably many horizontal edges, there are countably many even classes of orbits. Furthermore, since at most finitely many points in \overline{BC} are singular for each even class, there is a point O on \overline{BC} other than the midpoint that is nonsingular for every class. Therefore, given an even class $[\alpha]$, there is a point S and an element $x \in [\alpha]$ such that \overline{OS} is an unfolding of x . Note that if \overline{PQ} is an unfolding of α , then \overline{OS} is the horizontal translation of \overline{PQ} by $\overrightarrow{P\bar{O}}$. Therefore α uniquely determines the point S , denoted henceforth by S_α , and we refer to \overline{OS}_α as the *fundamental unfolding* of $[\alpha]$. The *fundamental region* at O , denoted by Γ_O , is the polar region $30^\circ \leq \theta \leq 60^\circ$ centered at O ; the points S_α are called *lattice points* of Γ_O .

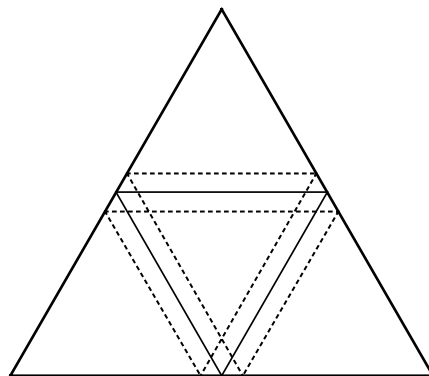


Figure 7. The Fagnano orbit and an equivalent period 6 orbit (dotted).

Since O is not the midpoint of \overline{BC} , odd iterates of Fagnano's orbit γ have no fundamental unfoldings. On the other hand, the fundamental unfolding of γ^{2n} represents the n -fold iterate of a primitive period 6 orbit. Nevertheless, with the notable exception of $[\gamma^2]$, "primitivity" is a property common to all orbits of the same class (see Figure 7). Indeed, the fundamental unfolding of $[\gamma^2]$ represents a primitive orbit. So we define a *primitive class* to be either $[\gamma^2]$ or a class of primitives.

To complete the classification, we must determine exactly which directed segments in Γ_O with initial point O represent orbits with even period. We address this question in the next section.

3. ORBITS AND RHOMBIC COORDINATES. In this section we introduce the analytical structure we need to complete the classification and to count the distinct classes of orbits of a given even period. Expressing a fundamental unfolding \underline{OS} as a vector \overrightarrow{OS} allows us to exploit the natural rhombic coordinate system given by \mathcal{T} . Let O be the origin and take the x -axis to be the horizontal line containing it. Take the y -axis to be the line through O with inclination 60° and let BC be the unit of length (see Figure 8). Then in rhombic coordinates

$$\Gamma_O = \{(x, y) \mid 0 \leq x \leq y\}.$$

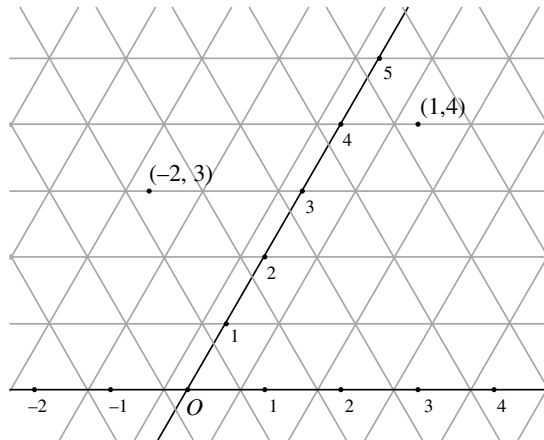


Figure 8. Rhombic coordinates.

Since the period of $[\alpha]$ is twice the number of rhombic tiles cut by OS_α , and the rhombic coordinates of S_α count these rhombic tiles, we can strengthen Theorem 1:

Corollary 3. *If $S_\alpha = (x, y)$, then α has period $2(x + y)$.*

Points in the integer sublattice \mathcal{L} of points on the horizontals of \mathcal{H} that are images of O under the action of G have the following simple characterization: Let H be the hexagon of \mathcal{H} with base \overline{BC} , and let τ_1 and τ_2 denote the translations by the vectors $(1, 1)$ and $(0, 3)$, respectively. Then the six hexagons adjacent to H are its images $\tau_2^b \tau_1^a(H)$, $(a, b) \in \{\pm(1, 0), \pm(1, -1), \pm(2, -1)\}$. Inductively, if H' is any hexagon of \mathcal{H} , then $H' = \tau_2^b \tau_1^a(H)$ for some $a, b \in \mathbb{Z}$. Note that $a(1, 1) + b(0, 3)$ defines the translation $\tau_2^b \tau_1^a$. Hence \mathcal{L} is generated by the vectors $(1, 1)$ and $(0, 3)$ and it follows that $(x, y) \in \mathcal{L}$ if and only if $x \equiv y \pmod{3}$.

Now recall that if PQ is an unfolding, then Q lies on a horizontal of \mathcal{H} . Hence OS is a fundamental unfolding if and only if $S \in \mathcal{L} \cap \Gamma_O - O$ if and only if $S \in \{(x, y) \in \mathbb{Z}^2 \cap \Gamma_O \mid x \equiv y \pmod{3}, x + y = n\}$. We have proved:

Theorem 3. *Given an even class $[\alpha]$, let $(x, y)_\alpha = S_\alpha$. The function*

$$\Phi : \{[\alpha] \mid [\alpha] \text{ has period } 2n\} \rightarrow \{(x, y) \in \mathbb{Z}^2 \cap \Gamma_O \mid x \equiv y \pmod{3}, x + y = n\}$$

given by $\Phi([\alpha]) = (x, y)_\alpha$ is a bijection.

Taken together, Proposition 3, Corollary 3, and Theorem 3 classify all periodic orbits on an equilateral triangle.

Theorem 4. (Classification) *Let α be a periodic orbit on an equilateral triangle.*

- (1) *If α has period $2n$, then $[\alpha]$ has cardinality c and contains exactly one representative whose unfolding OS satisfies $S = (x, y)$, $0 \leq x \leq y$, $x \equiv y \pmod{3}$, and $x + y = n$.*
- (2) *Otherwise, $\alpha = \gamma^{2k-1}$ for some $k \geq 1$, in which case its period is $6k - 3$.*

In view of Theorem 3, we may count classes of orbits of a given period $2n$ by counting integer pairs (x, y) such that $0 \leq x \leq y$, $x \equiv y \pmod{3}$ and $x + y = n$. This is the objective of the next and concluding section.

4. ORBITS AND INTEGER PARTITIONS. We will often refer to an ordered pair (x, y) as an “orbit” when we mean the even class of orbits to which it corresponds. Two questions arise: (1) Is there an orbit with period $2n$ for each $n \in \mathbb{N}$? (2) If so, exactly how many distinct classes of orbits with period $2n$ are there?

If we admit iterates, question (1) has an easy answer. Clearly there are no period 2 orbits since no two sides of $\triangle ABC$ are parallel—alternatively, if (a, b) is a solution of the system $x \equiv y \pmod{3}$ and $x + y = 1$, either a or b is negative. For each $n > 1$, the orbit

$$\alpha = \begin{cases} \left(\frac{n}{2}, \frac{n}{2}\right), & n \text{ even} \\ \left(\frac{n-1}{2} - 1, \frac{n-1}{2} + 2\right), & n \text{ odd.} \end{cases}$$

has period $2n$. Note that the period 22 orbits $(1, 10)$ and $(4, 7)$ are not equivalent since they have different lengths and representation angles (see Figures 9 and 10).

To answer question (2), we reduce the problem to counting partitions by constructing a bijection between classes of orbits with period $2n$ and partitions of n with 2 and 3 as parts. For a positive integer n , a *partition* of n is a nonincreasing sequence of nonnegative integers whose terms sum to n . Such a sequence has finitely many nonzero terms, called the *parts*. Thus, we seek pairs of nonnegative integers (a, b) such that $n = 2a + 3b$. The reader can easily prove:

Lemma 1. *For each $n \in \mathbb{N}$, let*

$$X_n = \{(a, b) \in \mathbb{Z}^2 \mid a, b \geq 0 \text{ and } 2a + 3b = n\} \text{ and} \\ Y_n = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq y, x \equiv y \pmod{3}, x + y = n\}.$$

The function $\varphi : X_n \rightarrow Y_n$ given by $\varphi(a, b) = (a, a + 3b)$ is a bijection.

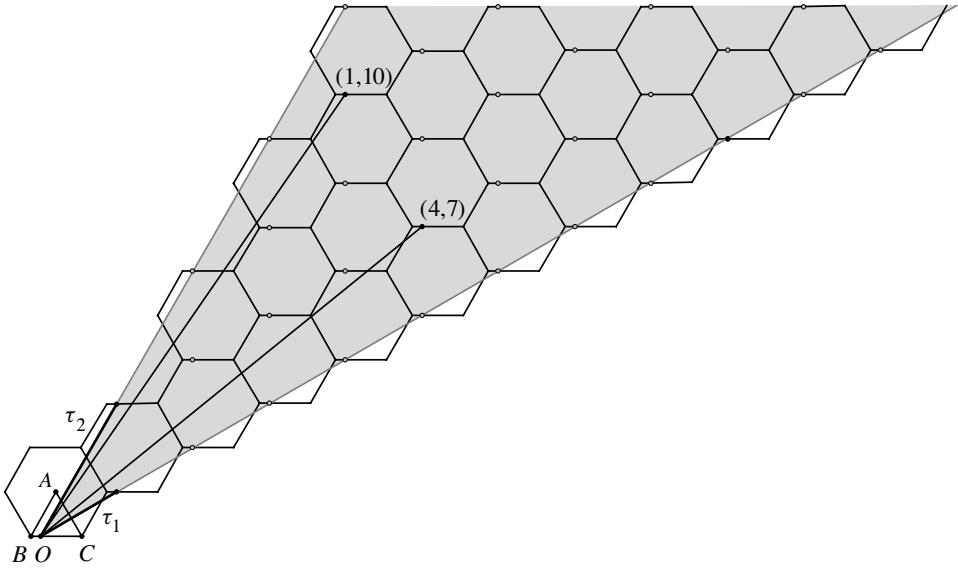


Figure 9. Translated images of O in Γ_O and unfoldings of period 22 orbits.

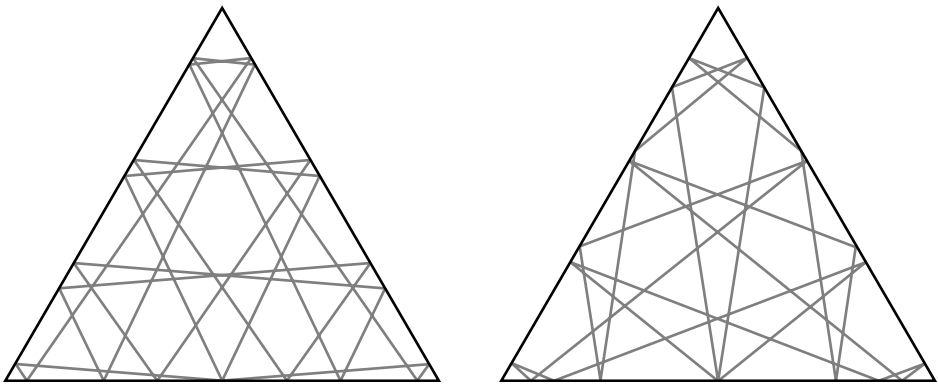


Figure 10. Period 22 orbits (1, 10) (left) and (4, 7) (right).

The composition $\varphi^{-1} \circ \Phi$ gives:

Corollary 4. For each $n \in \mathbb{N}$, there is a bijection between orbits of period $2n$ and partitions of n with 2 and 3 as parts.

Counting partitions of n with specified parts is well understood (e.g., Sloane's A103221, [12]). The number of partitions of n with 2 and 3 as parts is the coefficient of x^n in the generating function

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} \mathcal{O}(n)x^n \\
 &= (1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots) \\
 &= \frac{1}{(1 - x^2)(1 - x^3)}.
 \end{aligned}$$

To compute this coefficient, let ω be a primitive cube root of unity and perform a partial fractions decomposition. Then

$$\begin{aligned} f(x) &= \frac{1}{4(1+x)} + \frac{1}{4(1-x)} + \frac{1}{6(1-x)^2} + \frac{1}{9} \left(\frac{1+2\omega}{\omega-x} + \frac{1+2\omega^2}{\omega^2-x} \right) \\ &= \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n x^n + \frac{1}{4} \sum_{n=0}^{\infty} x^n + \frac{1}{6} \sum_{n=0}^{\infty} (n+1)x^n \\ &\quad + \frac{1}{9} \sum_{n=0}^{\infty} (\omega^{2n+2} + 2\omega^{2n} + \omega^{n+1} + 2\omega^n)x^n, \end{aligned}$$

and we have

$$\mathcal{O}(n) = \frac{(-1)^n}{4} + \frac{n}{6} + \frac{5}{12} + \frac{1}{9} (\omega^{2n+2} + 2\omega^{2n} + \omega^{n+1} + 2\omega^n).$$

Let $\lfloor x \rfloor$ denote the floor of x . By easy induction arguments, one can obtain the following simpler formulations (see [12]):

Theorem 5. *The number of distinct classes of period $2n$ is exactly*

$$\begin{aligned} \mathcal{O}(n) &= \begin{cases} \lfloor \frac{n}{6} \rfloor, & n \equiv 1 \pmod{6} \\ \lfloor \frac{n}{6} \rfloor + 1, & \text{otherwise} \end{cases} \\ &= \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor. \end{aligned}$$

Let us refine this counting formula by counting only primitives. For every divisor d of n , the (n/d) -fold iterate of a primitive period $2d$ orbit has period $2n$. Hence, if $\mathcal{P}(n)$ denotes the number of primitive classes of period $2n$, then

$$\mathcal{O}(n) = \sum_{d|n} \mathcal{P}(d).$$

A formula for $\mathcal{P}(n)$ is a direct consequence of the Möbius inversion formula (see [10]). The Möbius function $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ is defined by

$$\mu(d) = \begin{cases} 1, & d = 1 \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ for distinct primes } p_i \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 6. *For each $n \in \mathbb{N}$, there are exactly*

$$\mathcal{P}(n) = \sum_{d|n} \mu(d) \mathcal{O}(n/d)$$

primitive classes of period $2n$.

Theorems 5 and 6, together with Example 1 below, imply:

Corollary 5. $\mathcal{O}(n) = 0$ if and only if $n = 1$; $\mathcal{P}(n) = 0$ if and only if $n = 1, 4, 6, 10$.

Corollary 6. *The following are equivalent:*

- (1) *The integer n is 1 or prime.*
- (2) $\mathcal{P}(n) = \mathcal{O}(n)$.
- (3) *All classes of period $2n$ are primitive.*

Table 1 displays some values of \mathcal{O} and \mathcal{P} . The values $\mathcal{O}(4) = 1$, $\mathcal{P}(4) = 0$, and $\mathcal{P}(2) = 1$, for example, indicate that the single class of period 8 contains only 2-fold iterates of the primitive orbits in the single class of period 4.

Table 1. Sample Values for $\mathcal{O}(n)$ and $\mathcal{P}(n)$.

n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$	n	$2n$	$\mathcal{O}(n)$	$\mathcal{P}(n)$
1	2	0	0	31	62	5	5
2	4	1	1	32	64	6	3
3	6	1	1	33	66	6	3
4	8	1	0	34	68	6	2
5	10	1	1	35	70	6	4
6	12	2	0	36	72	7	2
7	14	1	1	37	74	6	6
8	16	2	1	38	76	7	3
9	18	2	1	39	78	7	4
10	20	2	0	40	80	7	2
11	22	2	2	41	82	7	7
12	24	3	1	42	84	8	2
13	26	2	2	43	86	7	7
14	28	3	1	44	88	8	4
15	30	3	1	45	90	8	4
16	32	3	1	46	92	8	3
17	34	3	3	47	94	8	8
18	36	4	1	48	96	9	3
19	38	3	3	49	98	8	7
20	40	4	2	50	100	9	4
21	42	4	2	51	102	9	5
22	44	4	1	52	104	9	4
23	46	4	4	53	106	9	9
24	48	5	1	54	108	10	3
25	50	4	3	55	110	9	6
26	52	5	2	56	112	10	4
27	54	5	3	57	114	10	6
28	56	5	2	58	116	10	4
29	58	5	5	59	118	10	10
30	60	6	2	60	120	11	2

We conclude with an example of a primitive class of period $2n$ for each $n \in \mathbb{N} - \{1, 4, 6, 10\}$. But first we need the following self-evident lemma:

Lemma 2. *Given an orbit $(x, y) \in \Gamma_{\mathcal{O}}$, let $d \in \mathbb{N}$ be the largest value such that $x/d \equiv y/d \pmod{3}$. Then (x, y) is primitive if and only if $d = 1$; otherwise (x, y) is a d -fold iterate of the primitive orbit $(x/d, y/d)$.*

This gives an easy check for primitivity.

Theorem 7. *An orbit $(x, y) \in \Gamma_O$ is primitive if and only if either*

- (1) $\gcd(x, y) = 1$ or
- (2) $(x, y) = (3a, 3b)$, $\gcd(a, b) = 1$, and $a \not\equiv b \pmod{3}$ for some $a, b \in \mathbb{N} \cup \{0\}$.

Proof. If $\gcd(x, y) = 1$, the orbit (x, y) is primitive. On the other hand, if $(x, y) = (3a, 3b)$, $a \not\equiv b \pmod{3}$, and $\gcd(a, b) = 1$ for some a, b , let d be as in Lemma 2. Then $d \neq 3$ since $a \not\equiv b \pmod{3}$. But $\gcd(a, b) = 1$ implies $d = 1$, so (x, y) is also primitive when (2) holds.

Conversely, given a primitive orbit (x, y) , let $c = \gcd(x, y)$. Then $cm = x \leq y = cn$ for some $m, n \in \mathbb{N} \cup \{0\}$; thus $m \leq n$, $\gcd(m, n) = 1$ and $cm \equiv cn \pmod{3}$. Suppose (2) fails. The reader can check that $3 \nmid c$, in which case $m \equiv n \pmod{3}$. But $x/c \equiv y/c \pmod{3}$ and the primitivity of (x, y) imply $c = 1$. ■

Example 1. Using Theorem 7, the reader can check that the following orbits of period $2n$ are primitive:

- $n = 2k + 1, k \geq 1 : (k - 1, k + 2)$
- $n = 2 : (1, 1)$
- $n = 4k + 4, k \geq 1 : (2k - 1, 2k + 5)$
- $n = 4k + 10, k \geq 1 : (2k - 1, 2k + 11)$.

Since $\mathcal{P}(n)$ tells us there are *no* primitive orbits of period 2, 8, 12 or 20, Example 1 exhibits a primitive orbit of every possible even period.

5. CONCLUDING REMARKS. Many interesting open questions remain; we mention three:

- (1) What can be said if the equivalence relation on the set of all periodic orbits defined above is defined more restrictively? For example, one could consider an equivalence relation in which equivalent orbits have cycles of incidence angles that differ by a *cyclic* permutation.
- (2) Every isosceles triangle admits a period 4 orbit resembling $(1, 1)$ and every acute triangle admits an orbit of period 6 resembling $(0, 3)$. Empirical evidence suggests that every acute isosceles triangle with base angle at least 54 degrees admits an orbit of period 10 resembling $(1, 4)$. Thus we ask: To what extent do the results above generalize to acute isosceles triangles?
- (3) Arbitrarily label the sides of the triangle 0, 1, 2 and consider the sequence of integers modulo 3 given by the successive bounces of a billiards trajectory. Clearly periodic trajectories yield periodic sequences. For example, the sequence 01020102... is given by the period 4 orbit $(1, 1)$. If $\{a_n\}$ is a periodic mod 3 sequence, is $\{a_n\}$ given by some billiards trajectory?

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ANDREW BAXTER received his B.A. in Mathematics from Millersville University in 2005. This article, his first published work, is based on his undergraduate research thesis, which won the MAA Eastern Pennsylvania and Delaware chapter Student Paper Competition in 2005. He is currently pursuing his Ph.D. in combinatorics at Rutgers University under the supervision of Doron Zeilberger. His wife Kristen is also pursuing her Ph.D. at Rutgers in Classical Studies.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway NJ 08854
Andrew.Baxter@gmail.com

RONALD UMBLE is a professor of mathematics at Millersville University of Pennsylvania, where he has been a faculty member since 1984. His research interests are in algebraic topology and the theory of operads and PROPs, in particular. He has directed numerous undergraduate research projects and has coauthored three published papers with students (including this one). He is a member of the Pennsylvania Zeta chapter (Temple University) of Pi Mu Epsilon and was the 1972 winner of the Zeta chapter's student paper competition.

Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551
ron.umble@millersville.edu