Journal of Mathematical Sciences, Vol. 152, No. 3, 2008

STRUCTURE RELATIONS IN SPECIAL A_∞-BIALGEBRAS

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Abstract. We compute the structure relations in special A_∞-bialgebras whose operations are limited to those defining the underlying A_∞-(co)algebra substructure. Such bialgebras appear as the homology of certain loop spaces. Whereas structure relations in general A_∞-bialgebras depend upon the combinatorics of permutahedra, only Stasheff's associahedra are required here.

CONTENTS

1. Introduction

A general A_∞-bialgebra is a DG-module (H, d) equipped with a family of structurally compatible operations $\omega_{j,i} : H^{\otimes i} \to H^{\otimes j}$, where $i, j \geq 1$ and $i + j \geq 3$ (see [6]). In *special* A_∞-bialgebras, $\omega_{j,i} = 0$ for $i, j \geq 2$ and the remaining operations $m_i = \omega_{1,i}$ and $\Delta_j = \omega_{j,1}$ define the underlying A_∞-(co)algebra substructure. Thus, special A_∞-bialgebras have the form $(H, d, m_i, \Delta_j)_{i,j\geq2}$ subject to the appropriate structure relations involving d, m_i, and Δ_j. These relations are much easier to describe than those in the general case, which require the S-U diagonal Δ_P on permutahedra. Instead, the S-U diagonal Δ_K on Stasheff's associahedra $K = \sqcup K_n$ is required here (see [5]).

A_∞-bialgebras are fundamentally important structures in algebra and topology. In general, the homology of every A_∞-bialgebra inherits an A_∞-bialgebra structure [7]; in particular, this holds for the integral homology of a loop space. In fact, over a field, the A_∞-bialgebra structure on the homology of a loop space specializes to the A_∞-(co)algebra structures observed by Gugenheim [2] and Kadeishvili [3].

The main result of this paper is the following simple formulation of the structure relations in special A_∞-bialgebras that do not involve d. Let TH denote the tensor module of H and let e^{n-2} denote the top-dimensional face of K_n. There is a "fraction product" on $M = \mathrm{End}(TH)$ (denoted here by "$\bullet$") and certain cellular cochains $\xi, \zeta \in C^*(K; M)$ such that for every $i, j \geq 2$,

$$\Delta_j \bullet m_i = \xi^j(e^{i-2}) \bullet \zeta^i(e^{j-2}),$$

where the exponents indicate certain Δ_K-cup powers.

Acknowledgment. I must acknowledge the fact that many of the ideas in this paper germinated during conversations with Samson Saneblidze, whose openness and encouragement led to this paper. For this, I express sincere thanks.

This research was partially supported by a Millersville University faculty research grant.

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 43, Topology and Its Applications, 2006.

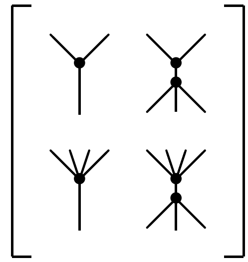
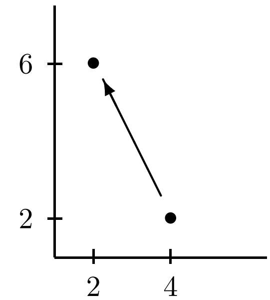
$A \in \mathbf{M}_{1,3}^{2,4} \quad \leftrightarrow$  $\leftrightarrow$ 

Fig. 1. Graphical representations of a typical monomial.

2. Matrix Considerations

We begin with a brief review of the algebraic machinery we need; for a detailed exposition, see [6]. Let $M = \bigoplus_{m,n\geq 1} M_{n,m}$ be a bigraded module over a commutative ring R with identity 1_R and consider the module TTM of tensors on TM. Given matrices $X = [x_{ij}]$ and $Y = [y_{ij}] \in \mathbb{N}^{q\times p}$, $p, q \geq 1$, consider the submodule

$$M_{Y,X} = \left(M_{y_{11},x_{11}} \otimes \cdots \otimes M_{y_{1p},x_{1p}}\right) \otimes \cdots \otimes \left(M_{y_{q1},x_{q1}} \otimes \cdots \otimes M_{y_{qp},x_{qp}}\right) \subset (M^{\otimes p})^{\otimes q} \subset TTM.$$

Represent a monomial $A = (\theta_{y_{11},x_{11}} \otimes \cdots \otimes \theta_{y_{1p},x_{1p}}) \otimes \cdots \otimes (\theta_{y_{q1},x_{q1}} \otimes \cdots \otimes \theta_{y_{qp},x_{qp}}) \in M_{Y,X}$ as a $(q \times p)$-matrix $[A] = [a_{ij}]$ with $a_{ij} = \theta_{y_{ij},x_{ij}}$. Then A is the q-fold tensor product of the rows of $[A]$ considered as elements of $M^{\otimes p}$; we refer to A as a $q \times p$ monomial and often write A instead of $[A]$. The *matrix submodule of TTM* is the sum

$$\overline{\mathbf{M}} = \bigoplus_{\substack{X,Y\in\mathbb{N}^{q\times p}\\ p,q\geq 1}} M_{Y,X} = \bigoplus_{p,q\geq 1} (M^{\otimes p})^{\otimes q}.$$

Given $\mathbf{x} \times \mathbf{y} = (x_1, \ldots, x_p) \times (y_1, \ldots, y_q) \in \mathbb{N}^p \times \mathbb{N}^q$, we set $X = [x_{ij} = x_j]_{1\leq i\leq q}$ and $Y = [y_{ij} = y_i]_{1\leq j\leq p}$ and denote $\mathbf{M}_{\mathbf{x}}^{\mathbf{y}} = M_{Y,X}$. The *bisequence submodule of TTM* is

$$\mathbf{M} = \bigoplus_{\substack{\mathbf{x}\times\mathbf{y}\in\mathbb{N}^p\times\mathbb{N}^q\\ p,q\geq 1}} \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$$

and the $(q \times p)$-monomial $A \in \mathbf{M}$ has the form

$$A = \begin{bmatrix} \theta_{y_1,x_1} & \cdots & \theta_{y_1,x_p} \\ \vdots & & \vdots \\ \theta_{y_q,x_1} & \cdots & \theta_{y_q,x_p} \end{bmatrix}.$$

We represent $A = [\theta_{y_j,x_i}] \in \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$ graphically in two ways: (1) as a matrix of "double corollas," where θ_{y_j,x_i} is pictured as two corollas joined at the root-one opening downward with x_i leaves and the other opening upward with y_j leaves and (2) as an arrow in the positive integer lattice $\mathbb{N}^2$ from $(|\mathbf{x}|, q)$ to $(p, |\mathbf{y}|)$, where $|\mathbf{u}| = u_1 + \cdots + u_k$ (see Fig. 1).

Each pairing $\gamma : \bigoplus_{r,s\geq 1} M^{\otimes r} \otimes M^{\otimes s} \to M$ induces an *upsilon product* $\Upsilon : \overline{\mathbf{M}} \otimes \overline{\mathbf{M}} \to \overline{\mathbf{M}}$ supported on "block transverse pairs," which we now describe.

Definition 1. A monomial pair $A^{q\times s} \otimes B^{t\times p} = [\theta_{y_{k\ell},v_{k\ell}}] \otimes [\eta_{u_{ij},x_{ij}}] \in \overline{\mathbf{M}} \otimes \overline{\mathbf{M}}$ is a

(i) *transverse pair* (TP) if $s = t = 1$, $u_{1,j} = q$, and $v_{k,1} = p$ for all j, k, i.e., setting $x_j = x_{1,j}$ and $y_k = y_{k,1}$ gives

$$A \otimes B = \begin{bmatrix} \theta_{y_1,p} \\ \vdots \\ \theta_{y_q,p} \end{bmatrix} \otimes \begin{bmatrix} \eta_{q,x_1} & \cdots & \eta_{q,x_p} \end{bmatrix} \in \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^{q};$$

(ii) *block transverse pair* (BTP) if there exist $(t \times s)$-block decompositions $A = [A'_{k'\ell}]$ and $B = [B'_{ij'}]$ such that $A'_{i\ell} \otimes B'_{i\ell}$ is a TP for *all* i, ℓ.

Unlike the blocks in a standard block matrix, the blocks $A'_{i\ell}$ (or $B'_{i\ell}$) in a general BTP may vary in length within a given row (or column). However, when $A \otimes B \in \mathbf{M}_{\mathbf{v}}^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^{\mathbf{u}}$ is a BTP with $\mathbf{u} = (q_1, \ldots q_t)$, $\mathbf{v} = (p_1, \ldots, p_s)$, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_s)$, and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_t)$, the TP $A'_{i\ell} \otimes B'_{i\ell} \in \mathbf{M}_{p_\ell}^{\mathbf{y}_i} \otimes \mathbf{M}_{\mathbf{x}_\ell}^{q_i}$ so that for a fixed i (or ℓ) the blocks $A'_{i\ell}$ (or $B'_{i\ell}$) have a constant length q_i (or p_ℓ); furthermore, $A \otimes B$ is a BTP if and only if $\mathbf{y} \in \mathbb{N}^{|\mathbf{u}|}$ and $\mathbf{x} \in \mathbb{N}^{|\mathbf{v}|}$ if and only if the initial point of the arrow A coincides with the terminal point of the arrow B. Note that BTP block decomposition is unique.

Example 1. A pairing of monomials $A^{4\times 2} \otimes B^{2\times 3} \in \mathbf{M}_{2,1}^{1,5,4,3} \otimes \mathbf{M}_{1,2,3}^{3,1}$ is a 2×2 BTP per the block decompositions

$$\begin{pmatrix} \begin{array}{c:c} \theta_{1,2} & \theta_{1,1} \\ \theta_{5,2} & \theta_{5,1} \\ \theta_{4,2} & \theta_{4,1} \\ \hdashline \theta_{3,2} & \theta_{3,1} \end{array} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \begin{array}{cc:c} \eta_{3,1} & \eta_{3,2} & \eta_{3,3} \\ \hdashline \eta_{1,1} & \eta_{1,2} & \eta_{1,3} \end{array} \end{pmatrix}.$$

Given a pairing $\gamma = \sum_{\mathbf{x}\times\mathbf{y}} \gamma_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^{q} \to \mathbf{M}_{|\mathbf{x}|}^{|\mathbf{y}|}$, extend γ to an *upsilon product* $\Upsilon : \overline{\mathbf{M}} \otimes \overline{\mathbf{M}} \to \overline{\mathbf{M}}$ via

$$\Upsilon(A \otimes B)_{i\ell} = \begin{cases} \gamma(A'_{i\ell} \otimes B'_{i\ell}) & \text{if } A \otimes B \text{ is a BTP} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then Υ sends a BTP $A^{q\times s} \otimes B^{t\times p} \in \mathbf{M}_{\mathbf{v}}^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^{\mathbf{u}}$ with $A'_{i\ell} \otimes B'_{i\ell} \in \mathbf{M}_{p_\ell}^{\mathbf{y}_i} \otimes \mathbf{M}_{\mathbf{x}_\ell}^{q_i}$ to a $(t \times s)$-monomial in $\mathbf{M}_{|\mathbf{x}_1|,\ldots,|\mathbf{x}_s|}^{|\mathbf{y}_1|,\ldots,|\mathbf{y}_t|}$. We denote $A \cdot B = \Upsilon(A \otimes B)$; if $[\theta_j] \otimes [\eta_i]$ is a TP, we denote $\gamma(\theta_1, \ldots, \theta_q; \eta_1, \ldots, \eta_p) = (\theta_1 \otimes \cdots \otimes \theta_q) \cdot (\eta_1 \otimes \cdots \otimes \eta_p)$. As an arrow, $A \cdot B$ runs from the initial point of B to the terminal point of A. Note that $\mathbf{M} \cdot \mathbf{M} \subseteq \mathbf{M}$, so that Υ is restricted to an upsilon product on $\mathbf{M}$.

Example 2. Continuing Example 1, the action of Υ on $A^{4\times 2} \otimes B^{2\times 3} \in \mathbf{M}_{2,1}^{1,5,4,3} \otimes \mathbf{M}_{1,2,3}^{3,1}$ produces a 2×2 monomial in $\mathbf{M}_{3,3}^{10,3}$:

$$\begin{pmatrix} \begin{array}{c:c} \theta_{1,2} & \theta_{1,1} \\ \theta_{5,2} & \theta_{5,1} \\ \theta_{4,2} & \theta_{4,1} \\ \hdashline \theta_{3,2} & \theta_{3,1} \end{array} \end{pmatrix} \cdot \begin{pmatrix} \begin{array}{cc:c} \eta_{3,1} & \eta_{3,2} & \eta_{3,3} \\ \hdashline \eta_{1,1} & \eta_{1,2} & \eta_{1,3} \end{array} \end{pmatrix} = \begin{pmatrix} \gamma(\theta_{1,2}, \theta_{5,2}, \theta_{4,2};\, \eta_{3,1}, \eta_{3,2}) & \gamma(\theta_{1,1}, \theta_{5,1}, \theta_{4,1};\, \eta_{3,3}) \\ \gamma(\theta_{3,2};\, \eta_{1,1}, \eta_{1,2}) & \gamma(\theta_{3,1};\, \eta_{1,3}) \end{pmatrix}.$$

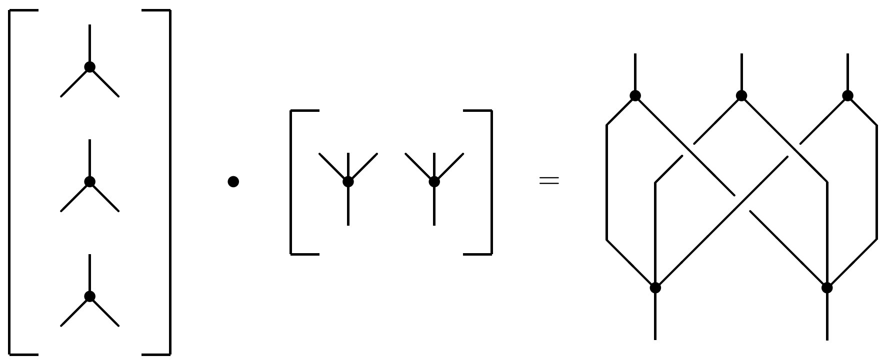


Fig. 2. The γ-product as a nonplanar graph.

In the target, $(|\mathbf{x}_1|, |\mathbf{x}_2|) = (1+2, 3)$ since $(p_1, p_2) = (2, 1)$ and $(|\mathbf{y}_1|, |\mathbf{y}_2|) = (1+5+4, 3)$ since $(q_1, q_2) = (3, 1)$. As an arrow, $A \cdot B$ initializes at $(6, 2)$ and terminates at $(2, 13)$.

The applications below relate to the following special case: let H be a graded module over a commutative ring with unity and consider $M = \operatorname{End}(TH)$ as a bigraded module via $M_{n,m} = \operatorname{Hom}(H^{\otimes m}, H^{\otimes n})$. Then a $q \times p$ monomial $A \in \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$ admits a representation as an operator on $\mathbb{N}^2$ via

$$(H^{\otimes|\mathbf{x}|})^{\otimes q} \approx (H^{\otimes x_1} \otimes \cdots \otimes H^{\otimes x_p})^{\otimes q} \xrightarrow{A} (H^{\otimes y_1})^{\otimes p} \otimes \cdots \otimes (H^{\otimes y_q})^{\otimes p} \xrightarrow{\sigma_{y_1,p} \otimes \cdots \otimes \sigma_{y_q,p}}$$

$$\xrightarrow{\sigma_{y_1,p} \otimes \cdots \otimes \sigma_{y_q,p}} (H^{\otimes p})^{\otimes y_1} \otimes \cdots \otimes (H^{\otimes p})^{\otimes y_q} \approx (H^{\otimes p})^{\otimes |\mathbf{y}|},$$

where $(s, t) \in \mathbb{N}^2$ is identified with $(H^{\otimes s})^{\otimes t}$ and $\sigma_{s,t} : (H^{\otimes s})^{\otimes t} \xrightarrow{\approx} (H^{\otimes t})^{\otimes s}$ is the canonical permutation of tensor factors

$$\sigma_{q,p} : \big((a_{11} \cdots a_{q1}) \cdots (a_{1p} \cdots a_{qp})\big) \mapsto \big((a_{11} \cdots a_{1p}) \cdots (a_{q1} \cdots a_{qp})\big).$$

The canonical structure mapping

$$\gamma = \sum \gamma_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \xrightarrow{\iota_p \otimes \iota_q} \mathbf{M}_{pq}^{|\mathbf{y}|} \otimes \mathbf{M}_{|\mathbf{x}|}^{qp} \xrightarrow{\operatorname{id} \otimes \sigma_{q,p}^*} \mathbf{M}_{pq}^{|\mathbf{y}|} \otimes \mathbf{M}_{|\mathbf{x}|}^{pq} \xrightarrow{\circ} \mathbf{M}_{|\mathbf{x}|}^{|\mathbf{y}|}, \tag{2.2}$$

where ι_p and ι_q are the canonical isomorphisms and $\sigma_{q,p}^*$ is induced by $\sigma_{q,p}$ (see [1, 4]), induces a canonical *associative* Υ product on $\mathbf{M}$ whose action on matrices of double corollas typically produces a matrix of nonplanar graphs (see Fig. 2).

In this setting, γ agrees with the composition product on the universal preCROC [8].

3. Cup Products

The two pairs of dual cup products defined in this section play an important role in the theory of structure relations. Let (H, dt) be a DG-module over a commutative ring with unity. For every $i, j \geq 2$, choose operations $m_i : H^{\otimes i} \to H$ and $\Delta_j : H \to H^{\otimes j}$ considered as elements of $M = \operatorname{End}(TH)$. Recall that planar rooted trees (PRTs) parameterize the faces of Stasheff's associahedra $K = \bigsqcup_{n \geq 2} K_n$ and provide module generators for cellular chains $C_*(K)$ [4]. Whereas top-dimensional faces correspond to corollas, lower-dimensional faces correspond to more general PRTs. Now, given a face $a \subseteq K$, consider the class of all planar rooted trees with levels (PLTs) representing a and choose a representative with exactly one node in each level. In this way, we obtain a particularly nice set of module generators for $C_*(K)$, denoted by $\mathcal{K}$. Note that the elements of a class of PLTs represent the same function obtained by the composition in various ways. The results obtained here are independent of the choice since they depend only on the function.

Let G be a DGA concentrated in degree zero and consider the cellular cochains on K with coefficients in G:

$$C^p(K;G) = \operatorname{Hom}^{-p}(C_p(K);G).$$

A diagonal Δ on $C_*(K)$ induces a cup product $\smile$ on $C^*(K;G)$ via

$$f \smile g = \cdot(f \otimes g)\Delta,$$

where "$\cdot$" denotes the multiplication in G.

The bisequence submodule $\mathbf{M}$, which serves as our coefficient module, is canonically endowed with dual associative *wedge* and *Čech cross products* defined on a monomial pair $A \otimes B \in \mathbf{M}_{\mathbf{v}}^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^{\mathbf{u}}$ as follows:

$$A \overset{\wedge}{\times} B = \begin{cases} A \otimes B & \text{if } \mathbf{v} = \mathbf{x}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad A \overset{\vee}{\times} B = \begin{cases} A \otimes B & \text{if } \mathbf{u} = \mathbf{y}, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $\overset{\wedge}{\mathbf{M}} = (\mathbf{M}, \overset{\wedge}{\times})$ and $\overset{\vee}{\mathbf{M}} = (\mathbf{M}, \overset{\vee}{\times})$ and note that $\mathbf{M}_{\mathbf{x}}^{\mathbf{y}} \overset{\wedge}{\times} \mathbf{M}_{\mathbf{x}}^{\mathbf{u}} \subseteq \mathbf{M}_{\mathbf{x}}^{\mathbf{y},\mathbf{u}}$ and $\mathbf{M}_{\mathbf{v}}^{\mathbf{y}} \overset{\vee}{\times} \mathbf{M}_{\mathbf{x}}^{\mathbf{y}} \subseteq \mathbf{M}_{\mathbf{v},\mathbf{x}}^{\mathbf{y}}$. Thus, nonzero cross-products concatenate matrices:

$$A \overset{\wedge}{\times} B = \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] \quad \text{and} \quad A \overset{\vee}{\times} B = [AB].$$

As arrows, $A \overset{\wedge}{\times} B$ runs from vertical $x = |\mathbf{x}|$ to vertical $x = p$, whereas $A \overset{\vee}{\times} B$ runs from horizontal $y = q$ to horizontal $y = |\mathbf{y}|$. In particular, if $A \in \mathbf{M}_a^b$, then $A^{\overset{\wedge}{\times} n} \in \mathbf{M}_a^{b \cdots b}$ is an arrow from (a, n) to $(1, nb)$ and $A^{\overset{\vee}{\times} n} \in \mathbf{M}_{a \cdots a}^b$ is an arrow from $(na, 1)$ to (n, b). These cross-products, together with the S-U diagonal Δ_K [5], induce wedge and Čech cup products $\wedge$ and $\vee$ in $C^*(K; \overset{\wedge}{\mathbf{M}})$ and $C^*(K; \overset{\vee}{\mathbf{M}})$, respectively.

The modules $C^*(K; \overset{\wedge}{\mathbf{M}})$ and $C^*(K; \overset{\vee}{\mathbf{M}})$ are equipped with second cup products $\wedge_\ell$ and $\vee_\ell$ arising from the Υ-product on $\mathbf{M}$ together with the "leaf coproduct" $\Delta_\ell : C_*(K) \to C_*(K) \otimes C_*(K)$, which we now define. Let $T = T^1 \in \mathcal{K}$ be a k-level PLT. Truncate T immediately below the first (top) level, trimming off a single corolla with n_1 leaves and $r_1 - 1$ stalks. Enumerating from left to right, let i_1 be the position of the corolla. The (*first*) *leaf sequence* of T is the r_1-tuple $\mathbf{x}_{i_1}(n_1) = (1 \cdots n_1 \cdots 1)$ with n_1 in position i_1 and 1 elsewhere. Denote the truncated tree by T^2; inductively, the jth *leaf sequence* of T is the leaf sequence of T^j. The induction terminates when $j = k$; in this case, $i_k = r_k = 1$ and $\mathbf{x}_{i_k}(n_k) = n_k$. The *descent sequence of* T is the k-tuple $(\mathbf{x}_{i_1}(n_1), \ldots, \mathbf{x}_{i_k}(n_k))$.

Definition 2. Let $T \in \mathcal{K}$ and identify T with its descent sequence $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_k)$. The *leaf coproduct* of T is given by

$$\Delta_\ell(T) = \begin{cases} \displaystyle\sum_{2 \le i \le k} (\mathbf{n}_1, \ldots, |\mathbf{n}_i|) \otimes (\mathbf{n}_i, \mathbf{n}_{i+1}, \ldots, \mathbf{n}_k), & k > 1 \\ 0, & k = 1. \end{cases}$$

Define the *leaf cup products* $\wedge_\ell$ and $\vee_\ell$ on $C^*(K; \overset{\wedge}{\mathbf{M}})$ and $C^*(K; \overset{\vee}{\mathbf{M}})$ as follows:

$$f \wedge_\ell g = \cdot(f \otimes g)\tau\Delta_\ell \text{ and } f \vee_\ell g = \cdot(f \otimes g)\Delta_\ell,$$

where τ interchanges tensor factors and $\cdot$ denotes the Υ-product.

Note that all cup products defined in this section are nonassociative and noncommutative. Unless otherwise explicitly indicated, iterated cup products are parenthesized on the extreme left, e.g., $f \vee g \vee h = (f \vee g) \vee h$.

4. Special A_∞-Bialgebras

Structural compatibility of d, m_i, and Δ_j is expressed in terms of the (restricted) biderivative d_ω and the "fraction product" $\bullet$ by the equation $d_\omega \bullet d_\omega = 0$. We begin with a construction of the biderivative

Fig. 3. The actions of φ and ψ.

in our restricted setting. Let $\varphi \in C^*(K; \hat{\mathbf{M}})$ and $\psi \in C^*(K; \check{\mathbf{M}})$ be the cochains with top-dimensional support such that

$$\varphi(e^{i-2}) = m_i \text{ and } \psi(e^{j-2}) = \Delta_j.$$

We consider φ and ψ as acting on up-rooted and down-rooted trees, respectively (see Fig. 3).

Let $T^c H$ denote the tensor coalgebra of H. The *coderivation cochain of* φ is the cochain $\varphi^c \in C^*(K; \hat{\mathbf{M}})$ that extends φ to cells of K in codim 1 so that

$$\sum_{\operatorname{codim} e = 0,1} \varphi^c(e) \in \operatorname{Coder}(T^c H)$$

is the co-free linear coextension of $\varphi(K) = \sum_{i \geq 2} \varphi(e^{i-2})$ as a coderivation. Thus, if $T \in \mathcal{K}$ is an uprooted 2-level tree with $n+k$ leaves and leaf sequence $\mathbf{x}_i(k)$, then

$$\varphi^c(T) = 1^{\otimes i-1} \otimes m_k \otimes 1^{\otimes n-i+1} = [1 \cdots m_k \cdots 1] \in \mathbf{M}^1_{\mathbf{x}_i(k)}$$

and is represented by the arrow from $(n+k, 1)$ to $(n+1, 1)$ on the horizontal axis in $\mathbb{N}^2$. Dually, let $T^a(H)$ denote the tensor algebra of H. The *derivation cochain of* ψ is the cochain $\psi^a \in C^*(K; \check{\mathbf{M}})$ that extends ψ to cells of K in codim 1 so that

$$\sum_{\operatorname{codim} e = 0,1} \psi^a(e) \in \operatorname{Der}(T^a H)$$

is the free linear extension of $\psi(K) = \sum_{i \geq 2} \psi(e^{i-2})$ as a derivation. Thus, if $T \in \mathcal{K}$ is a down-rooted 2-level tree with $n+k$ leaves and leaf sequence $\mathbf{y}_i(k)$, then

$$\psi^a(T) = 1^{\otimes i-1} \otimes \Delta_k \otimes 1^{\otimes n-i+1} = [1 \cdots \Delta_k \cdots 1]^T \in \mathbf{M}_1^{\mathbf{y}_i(k)}$$

and is represented by the arrow from $(1, n+1)$ to $(1, n+k)$ on the vertical axis.

Evaluating leaf cup powers of φ^c (resp. ψ^a) generates a representative of each class of compositions involving m_i (resp. Δ_j). Therefore, let

$$\xi = \varphi^c + \varphi^c \wedge_\ell \varphi^c + \cdots + (\varphi^c)^{\wedge_\ell k} + \cdots,$$
$$\zeta = \psi^a + \psi^a \vee_\ell \psi^a + \cdots + (\psi^a)^{\vee_\ell k} + \cdots,$$

and note that if e is a cell of K, each nonzero component of $\xi(e)$ (resp. $\zeta(e)$) is represented by a left-oriented horizontal (resp. upward-oriented vertical) arrow.

Furthermore, evaluating wedge and Čech cup powers of ξ (resp. ζ) generates the components of the co-free coextension of $\xi(K)$ as a Δ_K-coderivation (resp. free extension of $\zeta(K)$ as a Δ_K-derivation). Therefore, let

$$\hat{\varphi} = \xi + \xi \wedge \xi + \cdots + \xi^{\wedge k} + \cdots,$$
$$\check{\psi} = \zeta + \zeta \vee \zeta + \cdots + \zeta^{\vee k} + \cdots,$$

and note that the component $\xi^{\wedge k}(e^{i-2}) : (H^{\otimes i})^{\otimes k} \to (H^{\otimes 1})^{\otimes k}$ is represented by a left-oriented horizontal arrow from (i, k) to $(1, k)$ while the component $\zeta^{\vee k}(e^{i-2}) : (H^{\otimes 1})^{\otimes k} \to (H^{\otimes i})^{\otimes k}$ is represented by an upward-oriented vertical arrow from $(k, 1)$ to (k, i).

Let $M_0 = M_{1,1}$. For reasons soon to become clear, the only structure relations involving the differential d are the classical quadratic relations in an A_∞-(co)algebra. Note that $d \in M_0$ and let $\mathbf{1}^s = (1, \ldots, 1) \in \mathbb{N}^s$. Given $\theta \in M_0$ and $p, q \geq 1$, consider the monomials $\theta_i^{q\times 1} \in \mathbf{M}_{\mathbf{1}}^{\mathbf{1}^q}$ and $\theta_j^{1\times p} \in \mathbf{M}_{\mathbf{1}^p}^{1}$, all of whose entries are the identity except for the ith in $\theta_i^{q\times 1}$ and the jth in $\theta_j^{1\times p}$, both of which are θ. Define $\mathrm{Bd}_0 : M_0 \to \mathbf{M}$ by

$$\mathrm{Bd}_0(\theta) = \sum_{\substack{1\leq i\leq q,\, 1\leq j\leq p \\ p,q\geq 1}} \theta_i^{q\times 1} + \theta_j^{1\times p}.$$

Then $\mathrm{Bd}_0(\theta)$ is the (co)free linear (co)extension of θ as a (co)derivation. Note that each component of $\mathrm{Bd}_0(\theta)$ is represented by an arrow of "length" zero.

Let $M_1 = (M_{1,*} \oplus M_{*,1})/M_{1,1}$ and define $\mathrm{Bd}_1 : M_1 \to \mathbf{M}$ as follows:

$$\mathrm{Bd}_1(\theta) = \sum_{\substack{e\subseteq K \\ \operatorname{codim} e=0}} (\hat{\varphi} + \check{\psi})(e) + \sum_{\substack{e\subseteq K \\ \operatorname{codim} e=1}} (\varphi^c + \psi^a)(e). \tag{4.1}$$

Note that the components of $\mathrm{Bd}_1(\theta)$ are represented by upward-oriented vertical arrows and left-oriented horizontal arrows; the right-hand component of (4.1) is given by Gerstenhaber's $\circ$-(co)operation.

Let $\rho_0 : \mathbf{M} \to \mathbf{M}_0$ and $\rho_1 : \mathbf{M} \to \mathbf{M}_1$ denote the canonical projections.

Definition 3. The *restricted biderivative* is a (nonlinear) mapping $d_- : \mathbf{M} \to \mathbf{M}$ given by

$$d_- = \mathrm{Bd}_0 \circ \rho_0 + \mathrm{Bd}_1 \circ \rho_1.$$

The symbol d_θ denotes the restricted biderivative of θ.

Finally, the composition

$$\bullet : \mathbf{M} \times \mathbf{M} \overset{d_-\otimes d_-}{\longrightarrow} \mathbf{M} \times \mathbf{M} \overset{\Upsilon}{\longrightarrow} \mathbf{M}$$

defines the *fraction product*. Special A_∞-bialgebras are defined in terms of the fraction product as follows.

Definition 4. Let $\omega = d + \sum_{i,j\geq 2} (m_i + \Delta_j) \in M_0 \oplus M_1$. Then $(H, d, m_i, \Delta_j)_{i,j\geq 2}$ is a *special A_∞-bialgebra* if

$$d_\omega \bullet d_\omega = 0.$$

Note that one recovers the classical quadratic relations in an A_∞-algebra when $\omega = d + \sum_{i\geq 2} m_i$.

5. Structure Relations

The structure relations in a special A_∞-bialgebra $(H, d, m_i, \Delta_j)_{i,j\geq 2}$ easily follow from the following two observations:

(1) if $\theta, \eta \in \mathbf{M}$, then $\theta \bullet \eta = 0$ whenever the projection of θ or η to $M_0 \oplus M_1$ is zero;
(2) each nonzero component in the projections of θ and η is represented by a horizontal, vertical or zero-length arrow.

By (1), each component of $d_\omega \bullet d_\omega$ is a "transgression" represented by a "2-step" path of arrows from the horizontal axis $M_{1,*}$ to the vertical axis $M_{*,1}$ and by (2), each such 2-step path follows the edges of a (possibly degenerate) rectangle positioned with one of its vertices at $(1, 1)$.

Now relations involving d arise from degenerate rectangles since arrows of length zero represent components in the (co)extensions of d. Hence d interacts only with m_i or Δ_j and the relations involving d are exactly the classical quadratic relations in an A_∞-(co)algebra.

On the other hand, relations involving m_i and Δ_j arise from nondegenerate rectangles since m_i and Δ_j are represented by the arrows $(i, 1) \to (1, 1)$ and $(1, 1) \to (1, j)$. While the two-step path $(i, 1) \to (1, 1) \to$

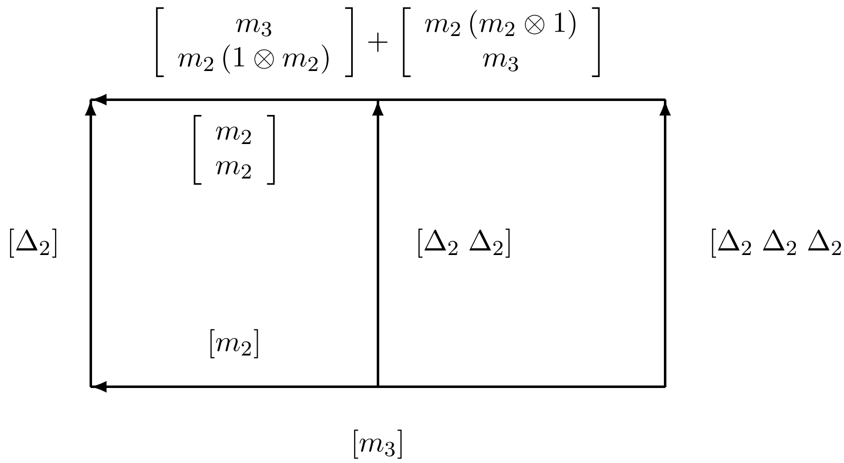


Fig. 4. Some low-order arrows in **M**.

$(1, j)$ represents the (usual) composition $\Delta_j \bullet m_i$, the two-step path $(i, 1) \to (i, j) \to (1, j)$ represents $\xi^j(e^{i-2}) \bullet \zeta^i(e^{j-2})$. Thus, we obtain the relation

$$\Delta_j \bullet m_i = \xi^j(e^{i-2}) \bullet \zeta^i(e^{j-2}).$$

For example, by setting $i = j = 2$ we obtain the classical bialgebra relation

$$\Delta_2 \bullet m_2 = \begin{bmatrix} m_2 \\ m_2 \end{bmatrix} \bullet [\Delta_2\Delta_2].$$

And for $(i, j) = (3, 2)$, we obtain

$$\Delta_2 \bullet m_3 = \left\{ \begin{bmatrix} m_3 \\ m_2(1\otimes m_2) \end{bmatrix} + \begin{bmatrix} m_2(m_2\otimes 1) \\ m_3 \end{bmatrix} \right\} \bullet [\Delta_2\Delta_2\Delta_2]$$

(see Fig. 4).

We summarize the discussion above in our main theorem.

Theorem 1. *Let $(H, d, m_i, \Delta_j)_{i,j\geq 2}$ be a special A_∞-bialgebra. Then $(H, d, m_i)_{i\geq 2}$ is an A_∞-algebra, $(H, d, \Delta_j)_{j\geq 2}$ is an A_∞-coalgebra and for all $i, j \geq 2$,*

$$\Delta_j \bullet m_i = \xi^j(e^{i-2}) \bullet \zeta^i(e^{j-2}).$$

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