A Non-operadic Operation on Loop Cohomology

Joint work with Samson Saneblidze

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The space X

- Denote $\bar{a}_i \in H^i\left(S^i;\mathbb{Z}_2\right)$, $i = 2, 3$, and $\bar{b} \in H^3\left(\Sigma\mathbb{C}P^2;\mathbb{Z}_2\right)$

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- Consider the following pullback of the path fibration:

$$\begin{array}{ccccc} K\left(\mathbb{Z}_2, 4\right) & \longrightarrow & X & \longrightarrow & \mathcal{L}K\left(\mathbb{Z}_2, 5\right) \\ & & p\downarrow & & \downarrow \\ & & \left(S^2 \times S^3\right) \vee \Sigma \mathbb{C}P^2 & \overset{f}{\longrightarrow} & K\left(\mathbb{Z}_2, 5\right) \\ & & \bar{a}_2 \bar{a}_3 + Sq^2 \bar{b} & \underset{f^*}{\longleftarrow} & \iota_5 \end{array}$$

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- Let $a_i = p^*\left(\bar{a}_i\right)$ and $b = p^*\left(\bar{b}\right)$ in $H^*\left(X;\mathbb{Z}_2\right)$

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- Let $a_i = p^*(\bar{a}_i)$ and $b = p^*(\bar{b})$ in $H^*(X; \mathbb{Z}_2)$
- $H^*(X; \mathbb{Z}_2) = \{1, a_2, a_3, b, a_2a_3 = Sq^2b, \ldots\}$

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- $(H, \Delta_H) \approx H^*(\Omega X; \mathbb{Z}_2)$ as graded coalgebras

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- The induced map $\phi : BA \otimes BA \longrightarrow A$ is given by

$$\phi([x] \otimes e) = \phi(e \otimes [x]) = x$$

$$\phi([b] \otimes [b]) = b \smile_1 b = a_2 a_3$$

and zero otherwise

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- Then $\mu\,([b] \otimes [b]) = [a_2 a_3]$

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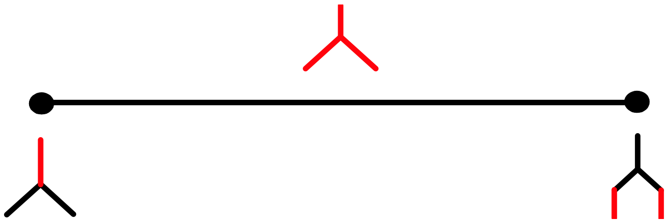
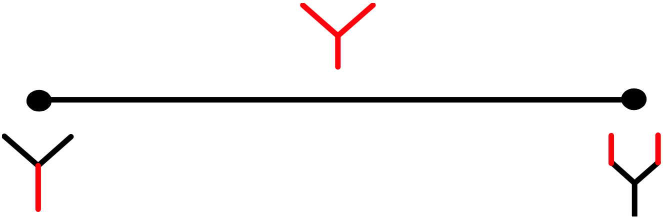
- The transfer process is controlled by *bimultiplihedra*

The Bimultiplihedron $JJ(1,2)$:



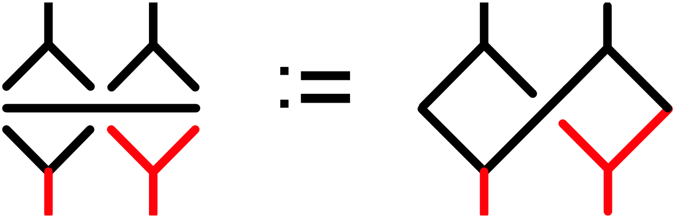
$$dg_2^1 = g\mu_H + \mu(g \otimes g)$$

The Bimultiplihedron $JJ(2,1)$:



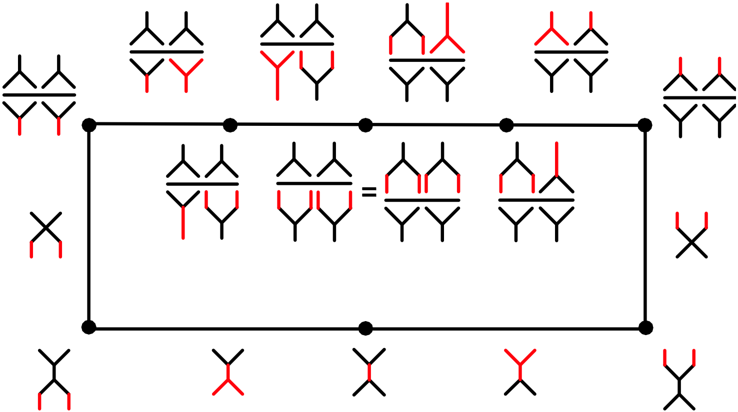
$$(d \otimes \mathbf{1} + \mathbf{1} \otimes d)\, g_1^2 = \Delta g + (g \otimes g)\, \Delta_H$$

Fraction notation

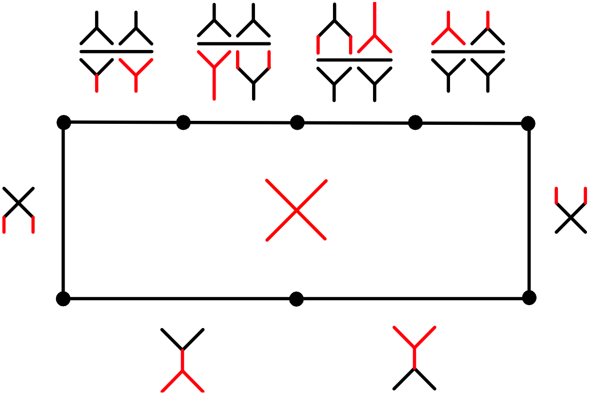


$$(\mu \otimes \mu)\, \sigma_{2,2}\, (\Delta g \otimes g_1^2)$$

The boundary of $JJ(2,2)$:



The $JJ(2,2)$ relation :



$$(d \otimes \mathbf{1} + \mathbf{1} \otimes d)\, g_2^2 = (\mu \otimes \mu)\, \sigma_{2,2} \left(\Delta g \otimes g_1^2 + g_1^2 \otimes (g \otimes g)\, \Delta_H \right) + \left(\mu\, (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H \right) \sigma_{2,2} \left(\Delta_H \otimes \Delta_H \right) + \omega_{BA}^{2,2} (g \otimes g) + (g \otimes g)\, \omega_H^{2,2} + \Delta g_2^1 + g_1^2 \mu_H$$

Transfer of structure

- Choose a cocycle-selecting map $g : H \longrightarrow BA$ such that

$$g\left(\mathrm{cls}\left[x_1|\cdots|x_n\right]\right) = \left[x_1|\cdots|x_n\right]$$

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- $\mu\left(\left[b\right] \otimes \left[b\right]\right) = \left[a_2 a_3\right] = d\left[a_2|a_3\right]$ implies

$$\mu_H\left(\beta \otimes \beta\right) = 0$$

g is homotopy multiplicative:

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- *The cochain homotopy g_2^1 satisfies*

$$g_2^1(\beta \otimes \beta) = [a_i|a_{5-i}] \text{ for some } i \in \{2,3\}$$

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$$g_1^2\,(\beta) = \lambda \otimes [a_2] + [a_2] \otimes \rho$$

  for some $\lambda, \rho \in \mathbb{Z}_2$

Non-triviality of the (2,2)-operation

- The Transfer Theorem implies that the $JJ_{2,2}$ relation holds:

$$
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(d \otimes \mathbf{1} + \mathbf{1} \otimes d)\, g_2^2 = {} & (\mu \otimes \mu)\, \sigma_{2,2}\left(\Delta g \otimes g_1^2 + g_1^2 \otimes (g \otimes g)\, \Delta_H\right) \\
& + \left(\mu\, (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H\right) \sigma_{2,2}\, (\Delta_H \otimes \Delta_H) \\
& + \omega_{BA}^{2,2}\, (g \otimes g) + (g \otimes g)\, \omega_H^{2,2} + \Delta g_2^1 + g_1^2 \mu_H
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- Let's evaluate the $JJ_{2,2}$ relation on $\beta\otimes\beta$

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$$= (\mu \otimes \mu)\, \sigma_{2,2} \left(\Delta [b] \otimes g_1^2 (\beta) + g_1^2 (\beta) \otimes \Delta [b]\right)$$

$$= 0 \text{ for every choice of } g_1^2 (\beta)$$

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- $\left(\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H\right) \sigma_{2,2} (\Delta_H \otimes \Delta_H) (\beta \otimes \beta)$

  $= \left(\mu (g \otimes g) \otimes g_2^1 + g_2^1 \otimes g \mu_H\right) \sigma_{2,2} (\Delta_H (\beta) \otimes \Delta_H (\beta))$

  $= 1 \otimes g_2^1 (\beta \otimes \beta) + g_2^1 (\beta \otimes \beta) \otimes 1$

  $= \left(\Delta + \overline{\Delta}\right) g_2^1 (\beta \otimes \beta)$

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- Thus the right-hand side of the $JJ_{2,2}$ relation reduces to

$$\left\{(g \otimes g)\,\omega_H^{2,2} + \left(\Delta + \overline{\Delta}\right) g_2^1 + \Delta g_2^1\right\}(\beta \otimes \beta)$$

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- But $\overline{\Delta} g_2^1\,(\beta\otimes\beta) = \overline{\Delta}\,[a_i|a_{5-i}] = [a_i]\otimes[a_{5-i}]$

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- But $\overline{\Delta} g_2^1 (\beta \otimes \beta) = \overline{\Delta} [a_i | a_{5-i}] = [a_i] \otimes [a_{5-i}]$

- Thus $(d \otimes \mathbf{1} + \mathbf{1} \otimes d)\, g_2^2 (\beta \otimes \beta)$

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Non-triviality of the (2,2)-operation

- Thus the right-hand side of the $JJ_{2,2}$ relation reduces to

$$\left\{(g\otimes g)\,\omega_H^{2,2}+\left(\Delta+\overline{\Delta}\right)g_2^1+\Delta g_2^1\right\}(\beta\otimes\beta)$$

$$=(g\otimes g)\,\omega_H^{2,2}(\beta\otimes\beta)+\overline{\Delta}g_2^1(\beta\otimes\beta)$$

- But $\overline{\Delta}g_2^1(\beta\otimes\beta)=\overline{\Delta}\,[a_i|a_{5-i}]=[a_i]\otimes[a_{5-i}]$

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$$=(g\otimes g)\,\omega_H^{2,2}(\beta\otimes\beta)+[a_i]\otimes[a_{5-i}]$$

- Since g is a cochain map, we conclude

$$\omega_H^{2,2}(\beta\otimes\beta)=\alpha_i\otimes\alpha_{5-i}$$

Summarizing...

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- **Conclusion:** *There is an induced non-operadic operation*

$$\omega_H^{2,2} : H \otimes H \to H \otimes H$$

of degree -1, *which is non-vanishing on* $\mathrm{cls}[b] \otimes \mathrm{cls}[b]$

Thank you!