Stamp Folding Puzzles:
A Delightful Excursion in Recreational Geometry

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In the 1930s, Stanislav Ulam posed the following **Map Folding Problem**: How many ways can one fold a sheet of square "stamps" into a packet the size of one stamp?

**Easier problem:** Suppose the "map" is a horizontal strip of stamps:

Number stamps from left-to-right. Fold with stamp #1 face up & upright.

Try this with a strip of three stamps.
A strip of three stamps can be folded six ways:

(Here the front of stamp #1 is marked)
Folding a Strip of \( n \) Stamps

\[
\begin{array}{cccccccccc}
\# \text{ Stamps:} & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots & n \\
\# \text{ Foldings:} & 2 & 6 & 16 & 50 & 144 & 462 & 1392 & 4536 & 14060 & \cdots & ?
\end{array}
\]

- No formula for the \( n^{th} \) term is known
Folding a Strip of $n$ Stamps

<table>
<thead>
<tr>
<th># Stamps:</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>...</th>
<th>$n$</th>
</tr>
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- On-line Encyclopedia of Integer Sequences: A000136
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- The Map Folding Problem is open and not well understood
Counting all possible foldings is a difficult problem...
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So instead, let’s fold the stamps in some specified configuration
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• So instead, let’s fold the stamps in some specified configuration

• Such problems are called **Stamp Folding Puzzles**
Stamp Folding Puzzle #1

Fold this $4 \times 4$ sheet of stamps into a $2 \times 2$ square showing the four green squares
Stamp Folding Puzzle #1

Fold this $4 \times 4$ sheet of stamps into a $2 \times 2$ square showing the four

- green squares
- yellow squares
Stamp Folding Puzzle #1

Fold this $4 \times 4$ sheet of stamps into a $2 \times 2$ square showing the four

- green squares
- yellow squares
- blue squares
Stamp Folding Puzzle #1

Fold this $4 \times 4$ sheet of stamps into a $2 \times 2$ square showing the four

- green squares
- yellow squares
- blue squares
- red squares
Stamp Folding Puzzle #2

Fold this block of equilateral triangular stamps into a packet 9-deep with stamps in the following order:

2 6 7 5 9 3 4 1 8

(Hint: tuck 5 between 7 and 9.)
Fold this block of isosceles right triangular stamps into a packet 16-deep with stamps in the following order:

4 1 16 6 5 15 14 8 7 13 11 12 2 3 9 10
Fold this block of 60°-right triangular stamps into a packet 12-deep with stamps in the following order:

5 2 8 9 7 3 4 11 12 1 6 10
Fredrickson’s Conjecture*

Although triangular stamps have come in a variety of different triangular shapes, only three shapes seem suitable for [stamp] folding puzzles: equilateral, isosceles right triangles, and 60°-right triangles.

The HYKU Theorem

(Hall, York, Kirby, U - 2009)

Exactly eight polygons generate edge tessellations of the plane:

- Equilateral triangle
- Isosceles triangle
- Right triangle
- Square
- Kite
- Regular hexagon
- Isosceles trapezoid
- Rectangle
Corollary (Settling Fredrickson’s Conjecture)

Exactly four shapes are suitable for stamp folding puzzles:

Rectangles; equilateral, isosceles right, $60^\circ$-right triangles.
Proof of Fredrickson’s Conjecture

- Let $G$ be a polygon generating a suitable edge tessellation
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- The interior angle of $G$ at $V$ has measure $\theta < 180^\circ$
Proof of Fredrickson’s Conjecture

- Let $G$ be a polygon generating a suitable edge tessellation.
- Let $V$ be a vertex of $G$.
- The interior angle of $G$ at $V$ has measure $\theta < 180^\circ$.
- Let $G'$ be the reflection of $G$ in an edge containing vertex $V$. 

![Diagram of polygon $G$ and its reflection $G'$]
The interior angle of $G'$ at $V$ has measure $\theta$; inductively, every interior angle at $V$ has measure $\theta$. 
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![Diagram showing angles $\theta$ and vertices $G$, $G'$, $V$]
The interior angle of $G'$ at $V$ has measure $\theta$; inductively, every interior angle at $V$ has measure $\theta$. 

\begin{center}
\begin{tikzpicture}
\node (V) at (0,0) {$V$};
\node (G) at (-2,-3) {$G$};
\node (G') at (2,-3) {$G'$};
\node (theta) at (-1,-1) {$\theta$};
\node (theta) at (1,-1) {$\theta$};
\end{tikzpicture}
\end{center}
A point $P$ is an $n$-center of a tessellation if the group of rotational symmetries centered at $P$ is generated by a rotation of $\phi_n = 360/n^\circ$.
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An $n$-center is even if $n$ is even.
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Successively reflecting in the edges of $G$ meeting at $V$ is a rotational symmetry.
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$V$ is an $n$-center for some $n > 1$
Admissible Interior Angles

- A point $P$ is an \textit{n-center} of a tessellation if the group of rotational symmetries centered at $P$ is generated by a rotation of $\phi_n = \frac{360}{n}\degree$

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- $V$ is an \textit{n-center} for some $n > 1$

$$\theta = \begin{cases} 
\phi_n & \text{if the bisector of } \angle V \text{ is a line of symmetry} \\
\frac{1}{2}\phi_n & \text{otherwise}
\end{cases}$$
Admissible Interior Angles

- **Case** $n = 2 : \phi_2 = 180^\circ$
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- $\theta < 180^\circ \Rightarrow \theta = 90^\circ$
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- Case $n \geq 4 : \phi_n \leq 90^\circ \Rightarrow \theta \in \left\{90^\circ, 72^\circ, 60^\circ, 51\frac{3}{7}^\circ, 45^\circ, \ldots\right\}$
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- **Conclude:** $\theta \in \{x \mid nx = 360^\circ, \ n \geq 3\}$
Admissible Interior Angles

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**Conclude:** $\theta \in \{x \mid nx = 360^\circ, \ n \geq 3\}$

**Obtuse interior angles measure** 120°
If $\theta = 120^\circ$, three copies of $G$ share vertex $V$. 
If $m\angle V = 120^\circ$, three copies of $G$ share vertex $V$.

Let $e$ and $e'$ be the edges of $G$ that meet at $V$, and labeled so that the angle from $e$ to $e'$ measures $120^\circ$. 

\[ V \]
Let $e'$ and $e''$ be the images of $e$ and $e'$ under a $120^\circ$ rotation.
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Then $e''$ and bisector $s$ of $\angle V$ are collinear.
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\begin{itemize}
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\end{itemize}
Angle Bisectors and Lines of Symmetry

Let $e'$ and $e''$ be the images of $e$ and $e'$ under a $120^\circ$ rotation.

Then $e''$ and bisector $s$ of $\angle V$ are collinear.

$e'$ is the reflection of $e$ in bisector $s$.

Bisector $s$ is on a line of symmetry.
Argument above depends only on the parity of $n$
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**Conclusion:** If $n$ is odd, the bisector $s$ of $\angle V$ is a line of symmetry
Angle Bisectors and Lines of Symmetry

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- **Conclusion**: *If $n$ is odd, the bisector $s$ of $\angle V$ is a line of symmetry*

- *A suitable edge tessellation has strictly even $n$-centers*
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**Conclusion:** If $n$ is odd, the bisector $s$ of $\angle V$ is a line of symmetry.

A suitable edge tessellation has strictly even $n$-centers.

Even $n$-centers $\Rightarrow \theta \in S = \{x \mid nx = 180^\circ, \ n \geq 2\}$

$$= \{90^\circ, 60^\circ, 45^\circ, 36^\circ, 30^\circ, \ldots\}$$
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**Conclusion:** If $n$ is odd, the bisector $s$ of $\angle V$ is a line of symmetry

A suitable edge tessellation has strictly even $n$-centers

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$$= \{90^\circ, 60^\circ, 45^\circ, 36^\circ, 30^\circ, \ldots\}$$

Only **non-obtuse** polygons generate suitable edge tessellations
Admissible Polygons

- Let $g$ be the number of edges of $G$
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- **Case** $g = 4$: Each interior angle $\theta_i \leq 90^\circ$
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- Interior angle sum of a quadrilateral: $360^\circ \Rightarrow \theta_i = 90^\circ$
Let $g$ be the number of edges of $G$

Interior angle sum: $180° (g - 2) \leq 90° g \Rightarrow g \leq 4$

**Case** $g = 4$: Each interior angle $\theta_i \leq 90°$

Interior angle sum of a quadrilateral: $360° \Rightarrow \theta_i = 90°$

**Conclusion:** $G$ is a rectangle
Admissible Right Triangles

- **Case** $g = 3 : G = \Delta ABC$
Admissible Right Triangles

- **Case** \( g = 3 \): \( G = \Delta ABC \)

- \( m\angle A \leq m\angle B \) and \( m\angle C = 90^\circ \)
Admissible Right Triangles

- **Case** $g = 3 : G = \triangle ABC$

- $m\angle A \leq m\angle B$ and $m\angle C = 90^\circ$

- $m\angle A + m\angle B = 90^\circ$
Admissible Right Triangles

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- \( m\angle A + m\angle B = 90^\circ \)

- \( (m\angle A, m\angle B) \in S \times S \Rightarrow \)

  \[
  (m\angle A, m\angle B) \in \{(30^\circ, 60^\circ), (45^\circ, 45^\circ)\}
  \]
Admissible Right Triangles

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  $$(m\angle A, m\angle B) \in \{(30^\circ, 60^\circ), (45^\circ, 45^\circ)\}$$

- **Conclusion:** $G$ is a $60^\circ$-right or an isosceles-right triangle
Each interior angle $\theta_i \leq 60^\circ$
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Interior angle sum of a triangle: $180^\circ \Rightarrow \theta_i = 60^\circ$
Admissible Acute Triangles

- Each interior angle $\theta_i \leq 60^\circ$

- Interior angle sum of a triangle: $180^\circ \Rightarrow \theta_i = 60^\circ$

- \textit{G is a equilateral triangle}
Admissible Acute Triangles

- Each interior angle $\theta_i \leq 60^\circ$
- Interior angle sum of a triangle: $180^\circ \Rightarrow \theta_i = 60^\circ$
- $G$ is a equilateral triangle
- The proof is complete
Recap:

Exactly four shapes are suitable for stamp folding puzzles:

Rectangles; equilateral, isosceles right, 60°-right triangles.
Thank you!