The Linear Algebra of Space-Time:

Length Contraction and Time Dilation Near the Speed of Light

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- Minkowski norm $\|(t, x)\| = \sqrt{t^2 - x^2}$ ranges over all non-negative real and positive imaginary values

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$$t^2 - x^2 = a^2 \tag{1}$$

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- $a \in \mathbb{R}^+$ defines a *real hyperbolic circle of radius a* (the hyperbola $t^2 - x^2 = a^2$ inside the light cone)
- $a = ib \in i\mathbb{R}^+$ defines an *imaginary hyperbolic circle of radius ib* (the hyperbola $x^2 - t^2 = b^2$ outside the light cone)

Three kinds of vectors

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- *Space-like vectors:* positive imaginary Minkowski norm; live outside the light-cone

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- Rotations fix circles centered at the origin and send lines through the origin to lines through the origin
- Reflections fix their reflecting lines pointwise

Matrix Representation

- Rotations represented by orthogonal matrices with determinant $+1$:

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- These form the *orthogonal group* $O(2)$, which has two components:

$$[\rho_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } [\rho_\theta]\,[\sigma_0] = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

The Orthogonal Group O(2)

- The component

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

parametrizes the circle

$$C_1 : u_1^2 + u_2^2 = 2$$

in the 2-plane spanned by

$$\mathcal{B}_1 = \left\{ \mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \right\}$$

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- The trivial rotation $[\rho_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{2}\mathbf{u}_1 + 0\mathbf{u}_2$ is on this circle

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- $\mathcal{B}_1 \cup \mathcal{B}_2$ linearly independent $\Rightarrow C_1 \cap C_2 = \varnothing$ and $C_1 \cup C_2 = O\left(2\right)$

Minkowski Isometries

- Hyperbolic rotations represented by the matrices

$$[R_\theta] = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

or in coordinates by

$$R_\theta\,(t, x) = (t\cosh\theta + x\sinh\theta, t\sinh\theta + x\cosh\theta)$$

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- R_θ fixes hyperbolic circles:

$$\left(\bar{t}\right)^2 - \left(\bar{x}\right)^2 = \left(t\cosh\theta + x\sinh\theta\right)^2 - \left(t\sinh\theta + x\cosh\theta\right)^2 = t^2 - x^2$$

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- R_θ sends lines through the origin to lines through the origin:

$$R_\theta(a,0) = a(\cosh\theta, \sinh\theta)$$

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- Reflections S_0 in the t-axis and S_∞ in the x-axis are represented by

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- $O(1,1)$ has four disjoint components:

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$$[R_\theta]\,[S_0]\,[S_\infty] = \begin{bmatrix} -\cosh\theta & -\sinh\theta \\ -\sinh\theta & -\cosh\theta \end{bmatrix}$$

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The components

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and

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form the two branches of the hyperbola

$$H_1 : u_1^2 - u_4^2 = 2$$

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- *The speed of light is the same in every frame of reference*

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- photon is a line on the light cone

Proper Ellapsed TIme

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- The *proper elapsed time* of P is: $\dfrac{s}{c} = t\sqrt{1 - v^2/c^2}$

- If P is at rest, for example, its proper elapsed time is t.

Lorentz Transformations

- Hyperbolic change of coordinates

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} c\bar{t} \\ \bar{x} \end{bmatrix}. \tag{2}$$

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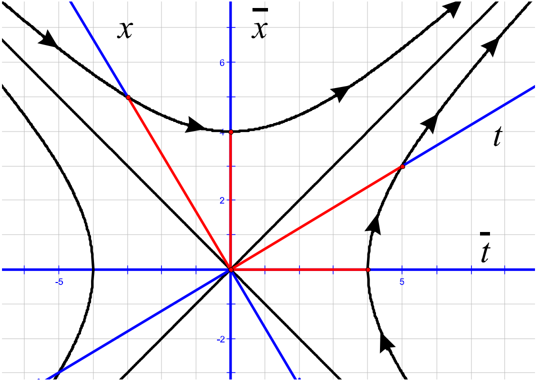
- $$\begin{aligned} ct &= c\bar{t}\cosh\theta + \bar{x}\sinh\theta \\ x &= c\bar{t}\sinh\theta + \bar{x}\cosh\theta \end{aligned}$$

Example

When $\theta = \ln 2$, $\cosh\theta = \frac{5}{4}$ and $\sinh\theta = \frac{3}{4}$

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$



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- The world line of a particle P at the origin of the moving frame $\bar{K}$ is the horizontal line $(c\bar{t}, 0)$

Lorentz Transformations

- *change coordinates of time-like vectors from $\bar{K}$ to K-coordinates*
- *change coordinates of space-like vectors from K to $\bar{K}$-coordinates*
- The world line of a particle P at the origin of the moving frame $\bar{K}$ is the horizontal line $(c\bar{t}, 0)$
- When viewed from frame K, this world line has positive slope and is parameterized by $(ct, x) = (c\bar{t}\cosh\theta, c\bar{t}\sinh\theta)$

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- Dividing second components of

  $$(ct, x) = (c\bar{t}\cosh\theta, c\bar{t}\sinh\theta)$$

  by first components gives

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- $1 = \cosh^2\theta - \sinh^2\theta = \cosh^2\theta\left(1 - \tanh^2\theta\right)$

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Lorentz Transformations

Substituting gives

$$t = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\bar{t} + \left(v/c^2\right)\bar{x}\right)$$

$$x = \frac{1}{\sqrt{1 - v^2/c^2}} \left(v\bar{t} + \bar{x}\right)$$

or in matrix form

$$\begin{bmatrix} t \\ x \end{bmatrix} = \frac{1}{\sqrt{1 - v^2/c^2}} \begin{bmatrix} 1 & v/c^2 \\ v & 1 \end{bmatrix} \begin{bmatrix} \bar{t} \\ \bar{x} \end{bmatrix} \tag{3}$$

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- Equivalently, $\Delta\bar{x} = \sqrt{1 - v^2/c^2}\,\Delta x$

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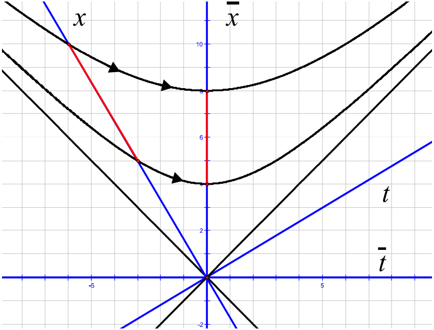
- If $\Delta x = \sqrt{34} \approx 5.83$, then $\Delta \bar{x} \approx (.69)\,(5.83) \approx 4$



Conclusion

To an observer in a reference frame moving along a straight line with constant speed v *relative to a fixed reference frame, the ordinary length of an object at rest in the fixed frame appears to be shorter than it does to an observer in the fixed frame by a factor of* $\sqrt{1 - v^2/c^2}$.

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- As the spaceship passes the space station, the captain takes two clock readings $\bar{t}_1$ and $\bar{t}_2$ and determines the elapsed time to be $\Delta\bar{t} = \bar{t}_2 - \bar{t}_1$
- Thinking of $\bar{t}_1$ and $\bar{t}_2$ as events $(\bar{t}_1, 0)$ and $(\bar{t}_2, 0)$, the relationship between $\Delta\bar{t}$ and Δt is

$$\Delta t = \frac{\bar{t}_2}{\sqrt{1 - v^2/c^2}} - \frac{\bar{t}_1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta\bar{t}$$

Example

- If $v = .73c$, then $\dfrac{1}{\sqrt{1 - .73^2}} \approx \dfrac{1}{\sqrt{.47}} \approx 1.46$

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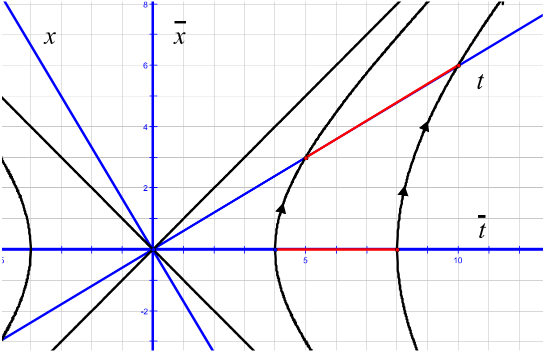
- If $v = .73c$, then $\dfrac{1}{\sqrt{1-.73^2}} \approx \dfrac{1}{\sqrt{.47}} \approx 1.46$
- If $\Delta\bar{t} = 4$, then $\Delta t \approx (1.46)\,(4) = 5.84$



Conclusion

To an observer in a fixed reference frame, the elapsed time measured in a reference frame moving along a straight line with constant speed v *appears to dilate by a factor of* $1/\sqrt{1 - v^2/c^2}$.

The End

Moral: Live fast; live long (relatively speaking...)!