

# The Saneblidze-Umble Diagonal on Associahedra

Presented by

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## The Permutahedron $P_n$

- $P_n = \text{convex hull} \{(\sigma(1), \dots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \in S_n\}$
- Vertices  $v_1, \dots, v_{n!}$  in the hyperplane  $x_1 + \dots + x_n = \binom{n}{2}$
- $P_n$  is an  $(n - 1)$ -dimensional convex polyhedron

## Combinatorics of $P_n$

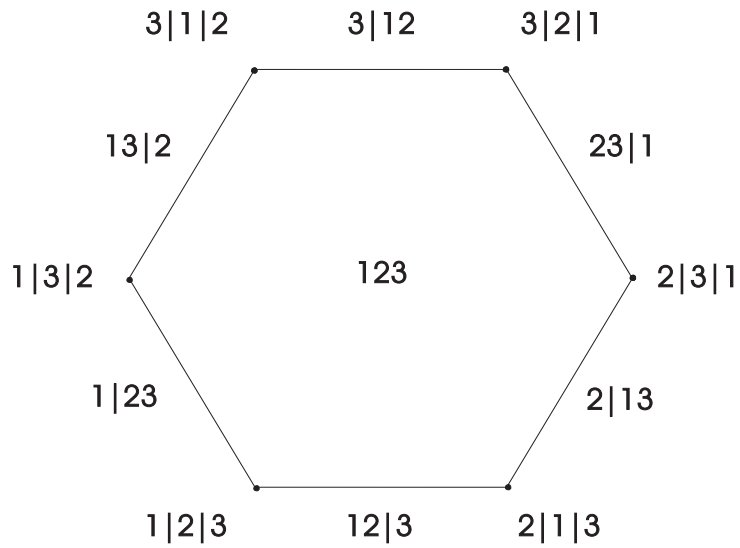
- Let  $\underline{n} = \{1, 2, \dots, n\}$
- $\{\text{Faces in codim } p\} \leftrightarrow \{\text{Partitions } U_1 \mid \dots \mid U_{p+1} \text{ of } \underline{n}\}$

$P_1$  is a point  $*$   $\leftrightarrow \{1\}$

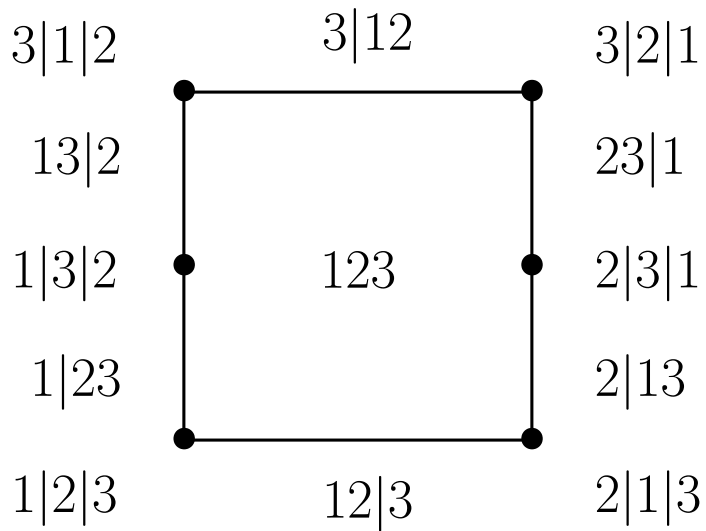
$P_2$  is a closed interval  $I$

edge  $\leftrightarrow \{12\}$ ; vertices  $\leftrightarrow \{1|2, 2|1\}$

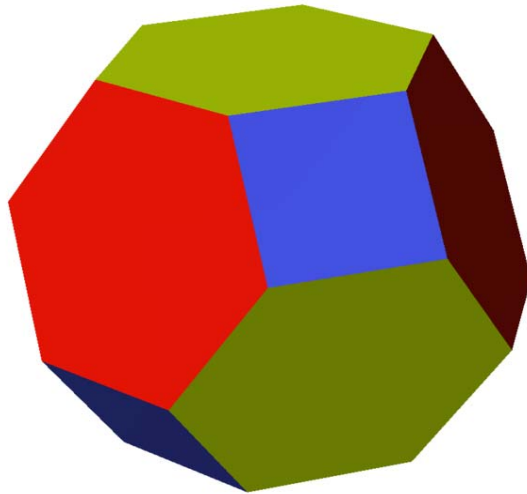
$P_3$  is a plane hexagonal region:



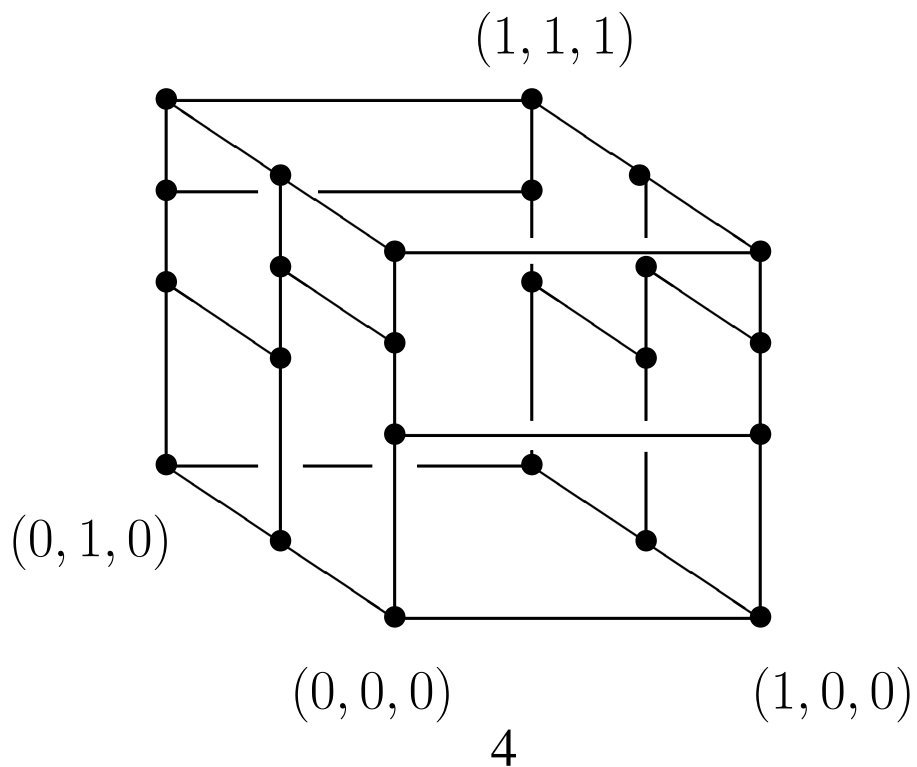
and can be obtained by subdividing  $P_2 \times I$ :



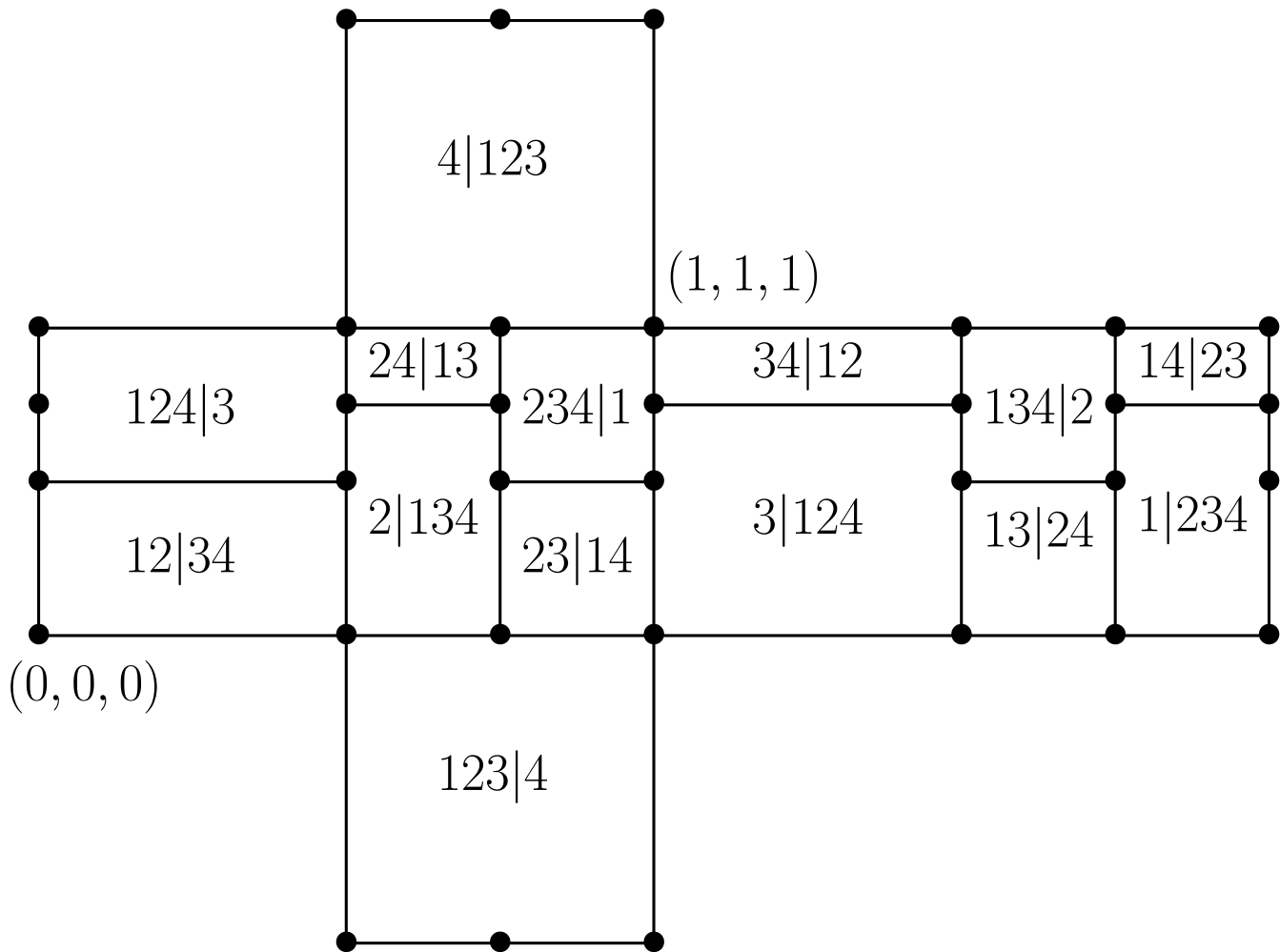
$P_4$  is a truncated octahedron:



and can be obtained by subdividing  $P_3 \times I$  :



The 2-faces of  $P_4$ :



Inductively,  $P_{n+1}$  can be obtained by subdividing  $P_n \times I$

## Cellular Chains

- Fix a ground field  $F$ ; let  $X$  be a cellular complex
- *Cellular  $k$ -chains* of  $X$  are elements of the  $F$ -vector space  $C_k(X)$  generated by the  $k$ -cells of  $X$
- The *graded vector space of cellular chains* of  $X$  is

$$C_*(X) = \bigoplus_{k \geq 0} C_k(X)$$

- *Cellular boundary*  $\partial$  is defined on the  $k$ -cells of  $X$  and extends to a linear map

$$\partial : C_*(X) \rightarrow C_{*-1}(X)$$

that satisfies  $\partial \circ \partial = 0$ .

- A *differential* on a graded vector space  $V_*$  is a linear map  $d : V_* \rightarrow V_{*-1}$  such that  $d \circ d = 0$
- $\partial$  is a differential on  $C_*(X)$

- $(V_*, d)$  is a *differential graded (DG) vector space*
- $(C_*(X), \partial)$  is DG vector space
- Let  $(V_*, d_V)$  and  $(W_*, d_W)$  be DG vector spaces.

A linear map  $f : V_* \rightarrow W_*$  is a *chain map* if

$$d_W \circ f = f \circ d_V$$

- Every map  $f : X \rightarrow Y$  of cellular complexes is homotopic to a cellular map  $g : X \rightarrow Y$ .
- A cellular map  $g : X \rightarrow Y$  induces a chain map  $g : C_*(X) \rightarrow C_*(Y)$  by restricting to the cells of  $X$  and extending linearly

## Diagonal Approximations

- The *geometric diagonal*  $\Delta : X \rightarrow X \times X$  is defined by  $\Delta(x) = x \times x$

- A *diagonal approximation* of  $\Delta$  is a cellular map  $\Delta_X : X \rightarrow X \times X$  homotopic to  $\Delta$

- Faces of the  $n$ -simplex  $\Delta^n = [012 \cdots n]$  are indexed by all (increasing) subsequences

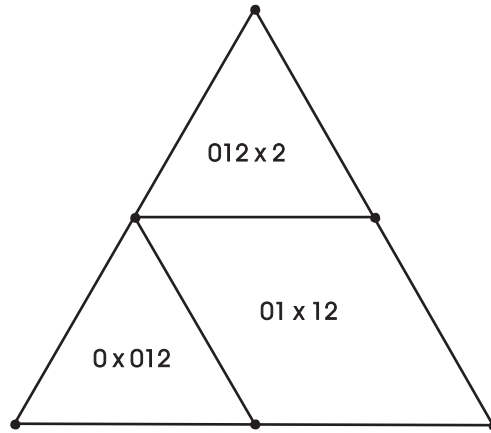
$$[a_0 \cdots a_k] \subseteq [012 \cdots n]$$

- $\partial([a_1 \cdots a_k]) = \sum (-1)^i [a_0 \cdots \hat{a}_i \cdots a_k]$

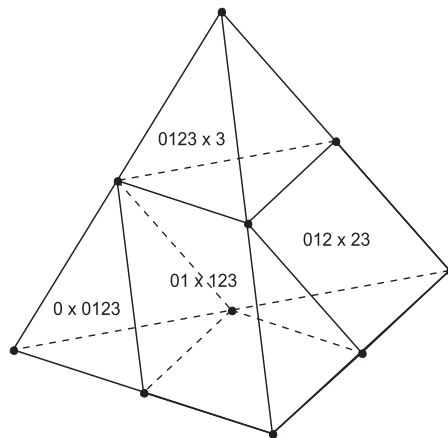
- Alexander-Whitney diagonal approximation:

$$\Delta_{\Delta^n}([012 \cdots n]) = \sum_{i=0}^n [01 \cdots i] \times [i \cdots n]$$

- A diagonal approx  $\Delta_X$  determines a tessellation of  $X$  that transforms  $\Delta_X$  into an inclusion



Tessellation of  $\Delta^2$  determined by A-W



Tessellation of  $\Delta^3$  determined by A-W

- Free linear extension of  $\partial$  to  $C_*(X) \otimes C_*(X)$  is a differential  $\partial \otimes 1 + 1 \otimes \partial$

- A *diagonal* on  $C_*(X)$  is a chain map

$$\Delta_X : C_*(X) \rightarrow C_*(X \times X) \approx C_*(X) \otimes C_*(X)$$

- A-W is strictly coassociative and homotopy cocommutative on cellular chains

## The Saneblidze-Umble Diagonal $\Delta_P$

- Step matrices

A matrix  $E$  is a *step matrix* if for some  $n$

- Each element of  $\underline{n}$  appears as an entry of  $E$  exactly once
- The elements of  $\underline{n}$  in each row and column of  $E$  form an increasing contiguous block
- Each diagonal parallel to the main diagonal of  $E$  contains exactly one element of  $\underline{n}$

A typical step matrix for  $n = 9$ :

$$E = \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & & & 5 \\ \hline & & 4 & 6 \\ \hline 1 & 3 & 8 & \\ \hline 7 & & & \\ \hline 9 & & & \\ \hline \end{array}$$

- Right-shift operator on  $G^{q \times p} = (g_{i,j})$

Let  $M_j \subset \{g_{*,j}\}$

If  $M_j \neq \emptyset$ , then  $\min M_j = g_{k,j}$  for some  $k$

If  $\min M_j > \max\{g_{*,j+1}\}$  and

$g_{t,j+1} = 0$  for  $k \leq t \leq q$ , obtain  $R_{M_j}G$

by interchanging  $g_{i,j} \in M_j$  and  $g_{i,j+1}$

Otherwise, define  $R_{M_j}G = G$

- Down-shift operator on  $G^{q \times p} = (g_{i,j})$

Let  $N_i \subset \{g_{i,*}\}$

If  $N_i = \emptyset$ , then  $\min N_i = g_{i,k}$  for some  $k$

If  $\min N_i > \max\{g_{i+1,*}\}$  and

$g_{i+1,t} = 0$  for  $k \leq t \leq p$ , obtain  $D_{N_i}G$

by interchanging  $g_{i,j} \in N_i$  and  $g_{i+1,j}$

Otherwise, define  $D_{N_i}G = G$ .

- Derived matrices

$$F = D_{N_p} D_{N_{p-1}} \cdots D_{N_1} R_{M_q} R_{M_{q-1}} \cdots R_{M_1} E$$

is a *derived matrix* iff  $E$  is a step matrix

- Step matrices are derived matrices via  $M_i = N_j = \emptyset$

A typical derived matrix:

$$E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad \rightarrow \quad D_{\emptyset} D_{\emptyset} R_5 R_{\emptyset} E = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & & 5 \\ \hline 4 & & \\ \hline \end{array}$$

- Complementary pairs

$A_1|A_2|\cdots|A_p \times B_q|B_{q-1}|\cdots|B_1$  is a

*complementary pair* (CP) of partitions iff  $B_i$

and  $A_j$  are the rows and columns of a  $q \times p$

derived matrix

- The CP  $14|25|3 \times 4|15|23 \leftrightarrow E$  above

- $\{\text{Derived matrices}\} \leftrightarrow \{\text{CPs}\}$
- $\{\text{CPs}\} \leftrightarrow \{\text{Faces of } P_n \times P_n\}$
- The S-U diagonal  $\Delta_P$  (2004)

Let  $e^{n-1}$  be the top dim'l face of  $P_n$

Define  $\Delta_P(e^0) = e^0 \otimes e^0$ .

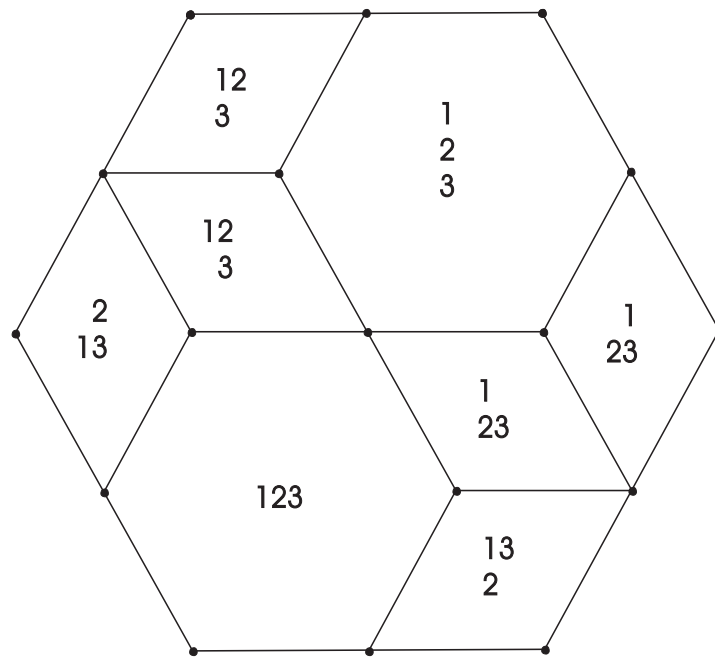
Having defined  $\Delta_P$  on  $C_*(P_k)$ ,  $k \leq n - 1$ ,

define  $\Delta_P$  on  $C_{n-1}(P_n)$  by

$$\Delta_P(e^{n-1}) = \sum_{\substack{(p,q)\text{-CPs } u \times v \\ p+q=n+1}} \pm u \otimes v$$

and extend multiplicatively to  $C_*(P_n)$

$\Delta_P(P_3)$  :



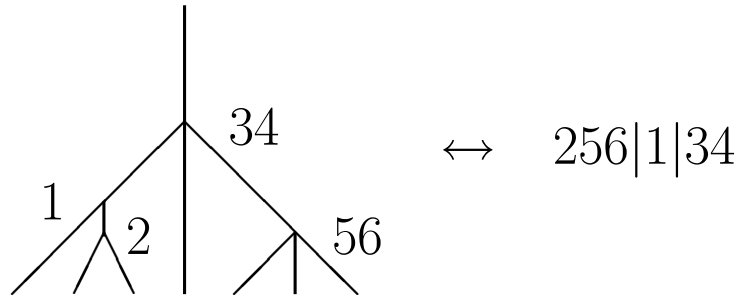
## The Associahedron $K_n$ (Stasheff 1963)

- $K_n = P_{n-1} / \sim$
- $\{\text{Codim } p \text{ faces of } P_{n-1}\} \leftrightarrow$   
 $\{\text{PLTs with } n \text{ leaves and } p + 1 \text{ levels}\}$   
(PLT = Planar rooted Leveled Tree)
- $\{\text{PLTs with } n \text{ leaves and } p + 1 \text{ levels}\} \leftrightarrow$   
 $\{\text{Partitions of } \underline{n - 1} \text{ of length } p + 1\}$

Given a PLT  $T$ , number the leaves from left to right and assign the label  $i$  to the node at which the branch of leaf  $i$  meets the branch of leaf  $i + 1$ . Then

$$T \leftrightarrow U_1 | \cdots | U_{p+1},$$

where  $U_j = \{\text{labels of nodes in level } j\}$



- $\{\text{Faces of } K_n\} \leftrightarrow \{\text{PRTs with } n \text{ leaves}\}$

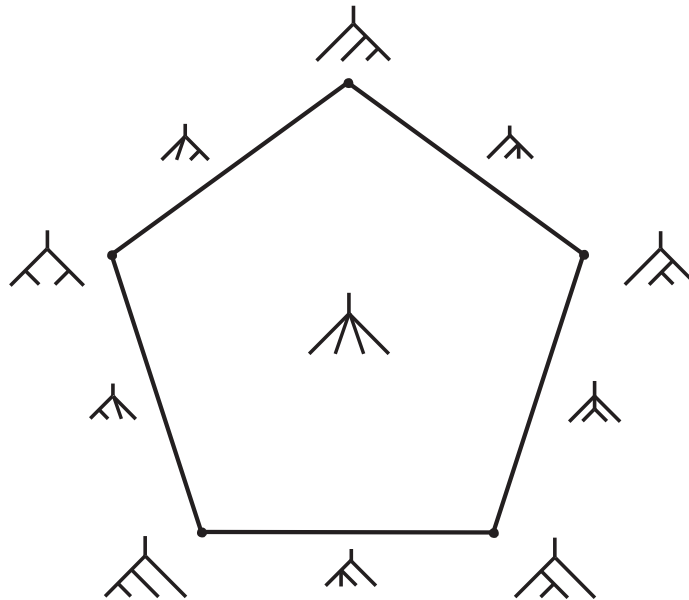
(PRT = Planar Rooted Tree)

$K_2$  is a point  $*$   $\leftrightarrow$   $\{\text{ } \wedge \text{ }\}$

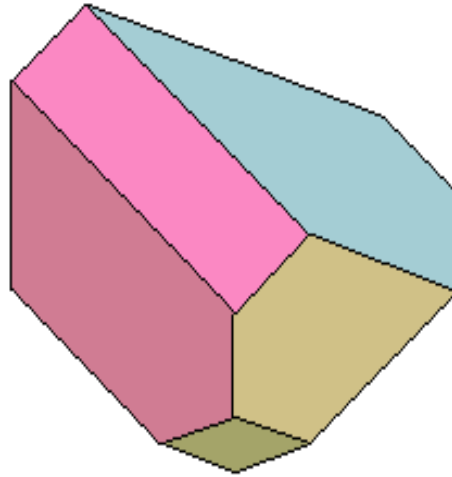
$K_3$  is a closed interval  $I$

edge  $\leftrightarrow$   $\{\text{ } \wedge \text{ }\}$ ; vertices  $\leftrightarrow$   $\{\text{ } \wedge \text{ }, \text{ } \wedge \text{ }\}$

$K_4$  is a plane pentagonal region:



$K_5$ :



- $\{\text{PRTs with } n \text{ leaves}\} \leftrightarrow$

$\{\text{Parenthesizations of } n \text{ variables}\}$

- $\partial : C_*(K_n) \rightarrow C_{*-1}(K_n)$  is defined by

$$\partial(T) = \sum \pm T_i,$$

where  $T_i$  has one more node than  $T$  and the  $T_i$ 's range over all possible trees obtained from  $T$  by grafting in a branch at some node of valance  $> 3$

$$\partial(\text{A}) = \text{A} - \text{A}$$

$$\partial(\text{A}) = \text{A} + \text{A} + \text{A} - \text{A} - \text{A}$$

- Forgetting levels defines Tonks' projection (1997)

$$\theta : P_{n-1} \rightarrow K_n$$

- Faces of  $P_{n-1}$  indexed by PLTs with multiple nodes in the same level degenerate under  $\theta$ ; corresponding generators lie in the kernel of the induced map  $\theta : C_*(P_{n-1}) \rightarrow C_*(K_n)$

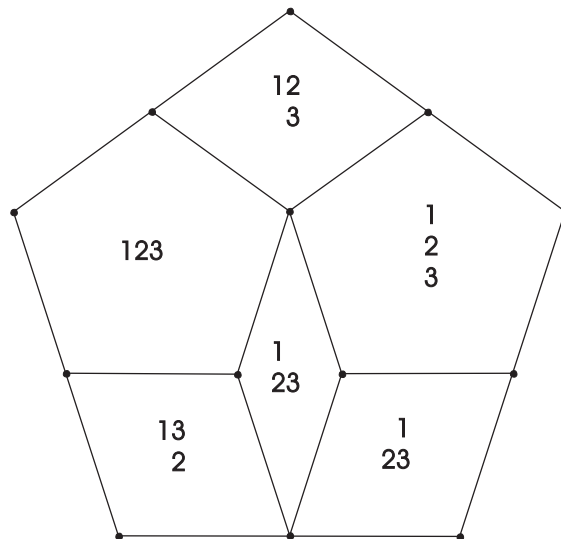
- The S-U diagonal  $\Delta_K$  is defined by

$$\Delta_K \theta = (\theta \otimes \theta) \Delta_P$$

- $\Delta_K(\text{Y}) = \text{Y} \otimes \text{Y}$

- $\Delta_K(\text{Y}) = \text{Y} \otimes \text{Y} + \text{Y} \otimes \text{Y}$

- $\Delta_K(\text{Y}) = \text{Y} \otimes \text{Y} + \text{Y} \otimes \text{Y} + \text{Y} \otimes \text{Y}$   
 $+ \text{Y} \otimes \text{Y} + \text{Y} \otimes \text{Y} + \text{Y} \otimes \text{Y}$



## $A_\infty$ -algebras (Stasheff 1963)

- Let  $(A, d)$  be a DG vector space and consider

$$\{ \text{Hom}^p (A^{\otimes n}, A) \}_{p \in \mathbb{Z}}$$

- Free linear extension of  $d$  to  $A^{\otimes n}$  is a differential

$$d_n = d \otimes 1^{\otimes n-1} + 1 \otimes d \otimes 1^{n-2} + \dots + 1^{\otimes n-1} \otimes d$$

- $d$  induces a differential

$$\delta : \text{Hom}^p (A^{\otimes n}, A) \rightarrow \text{Hom}^{p-1} (A^{\otimes n}, A) \quad \text{via}$$

$$\delta (f) = d \circ f - (-1)^p f \circ d_n$$

- $(\text{Hom}^* (A^{\otimes n}, A), \delta)$  is a DG vector space

- Identify the top dimensional cell  $e^{n-2}$  of  $K_n$  with a multilinear operation  $\mu_n \in \text{Hom}^{n-2}(A^{\otimes n}, A)$
- An  $A_\infty$ -algebra is a DG vector space  $(A, d)$  together with a family of operations

$$\{\mu_n \in \text{Hom}^{n-2}(A^{\otimes n}, A)\}_{n \geq 2}$$

and a family of chain maps

$$\{\zeta : C_*(K_n) \rightarrow \text{Hom}^{n-2}(A^{\otimes n}, A)\}_{n \geq 2}$$

such that  $\zeta(e^{n-2}) = \mu_n$  and  $\zeta(T)$  on a proper face  $T \subset K_n$  is the composition of  $\mu_k$ 's specified by  $T$

- Quadratic Structure Relations

Multiplication is homotopy associative and  $\mu_3$  is an associating homotopy:

$$\begin{aligned}
 \delta\mu_3 &= \delta\zeta(\text{⋈}) = \zeta\partial(\text{⋈}) \\
 &= \zeta(\text{⋈} - \text{⋈}) \\
 &= \mu_2(1 \otimes \mu_2) - \mu_2(\mu_2 \otimes 1)
 \end{aligned}$$

Stasheff's Pentagon Condition:

$$\begin{aligned}
 \delta\mu_4 &= \delta\zeta(\text{⋈}) = \zeta\partial(\text{⋈}) \\
 &= \zeta(\text{⋈} + \text{⋈} + \text{⋈} - \text{⋈} - \text{⋈}) \\
 &= \mu_2(\mu_3 \otimes 1) + \mu_3(1 \otimes \mu_2 \otimes 1) \\
 &\quad + \mu_2(1 \otimes \mu_3) - \mu_3(\mu_2 \otimes 1 \otimes 1) \\
 &\quad - \mu_3(1 \otimes 1 \otimes \mu_2)
 \end{aligned}$$

## Tensor Products of $A_\infty$ -algebras (S-U 2000)

- Let  $(A, \zeta_A)$  and  $(B, \zeta_B)$  be  $A_\infty$ -algebras.

The *induced  $A_\infty$ -algebra structure*  $\zeta_{A \otimes B}$  is given by the composition

$$\begin{array}{ccc}
 C_*(K_n) & \xrightarrow{\zeta_{A \otimes B}} & \text{Hom}((A \otimes B)^{\otimes n}, A \otimes B) \\
 \Delta_K \downarrow & & \uparrow (\sigma_{2,n})^*
 \end{array}$$

$$C_*(K_n) \otimes C_*(K_n) \xrightarrow[\zeta_A \otimes \zeta_B]{} \text{Hom}(A^{\otimes n} \otimes B^{\otimes n}, A \otimes B)$$

where  $\sigma_{2,n} : (A \otimes B)^{\otimes n} \rightarrow A^{\otimes n} \otimes B^{\otimes n}$  is the canonical permutation of tensor factors

- The  $A_\infty$ -algebra operations on  $A \otimes B$  are

$$\Phi_n = \zeta_{A \otimes B} (e^{n-2}) = [(\zeta_A \otimes \zeta_B) \Delta_K (e^{n-2})] \sigma_{2,n}$$

and in particular

$$\Phi_2 = [(\zeta_A \otimes \zeta_B) (\wedge \otimes \wedge)] \sigma_{2,2}$$

$$\Phi_3 = [(\zeta_A \otimes \zeta_B) (\wedge \otimes \wedge + \wedge \otimes \wedge)] \sigma_{2,3}$$

$$\begin{aligned} \Phi_4 = [(\zeta_A \otimes \zeta_B) (\wedge \otimes \wedge + \wedge \otimes \wedge \\ + \wedge \otimes \wedge + \wedge \otimes \wedge + \wedge \otimes \wedge \\ + \wedge \otimes \wedge)] \sigma_{2,4} \end{aligned}$$

⋮

⋮

## Tensor Products of Simple $A_\infty$ -algebras

- A simple  $A_\infty$ -algebra has exactly one non-trivial operation  $\mu_n$  for  $n \geq 3$

- Let  $A = \Lambda(x, y)$  with  $|x| = 1$ ,  $|y| = 4$  and  $d = 0$

Define

$$\mu_3(x^i y^p | x^j y^q | x^k y^r) = \begin{cases} y^{p+q+r+1}, & ijk = 1 \text{ and } pqr \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

All quadratic relations hold:

$$\begin{aligned} \mu_2(\mu_3 \otimes 1) + \mu_3(1 \otimes \mu_2 \otimes 1) + \mu_2(1 \otimes \mu_3) \\ = \mu_3(\mu_2 \otimes 1 \otimes 1) + \mu_3(1 \otimes 1 \otimes \mu_2) \end{aligned}$$

$$\mu_3(\mu_3 \otimes 1 \otimes 1 + 1 \otimes \mu_3 \otimes 1 + 1 \otimes 1 \otimes \mu_3) = 0$$

Therefore  $(A, \mu, \mu_3)$  is a simple  $A_\infty$ -algebra

- Over  $\mathbb{Z}_3$  there are two non-trivial higher order operations on  $A \otimes A$ , namely,

$$\Phi_3 = [\mu_2(\mu_2 \otimes 1) \otimes \mu_3 + \mu_3 \otimes \mu_2(1 \otimes \mu_2)] \sigma_{2,3}$$

$$\begin{aligned} \Phi_4 = & [\mu_2(\mu_3 \otimes 1) \otimes \mu_3(1 \otimes \mu_2 \otimes 1) \\ & + \mu_2(\mu_3 \otimes 1) \otimes \mu_2(1 \otimes \mu_3) \\ & + \mu_3(1 \otimes \mu_2 \otimes 1) \otimes \mu_2(1 \otimes \mu_3) \\ & + \mu_3(\mu_2 \otimes 1 \otimes 1) \otimes \mu_3(1 \otimes 1 \otimes \mu_2)] \sigma_{2,4}. \end{aligned}$$

## Application to Group Cohomology

- $H^*(C_n; \mathbb{Z}_p)$  is an  $A_\infty$ -algebra (Madson 2002)
- The  $A_\infty$ -algebra structure of

$$H^*(C_m \times C_n; \mathbb{Z}_p) \approx H^*(C_m; \mathbb{Z}_p) \otimes H^*(C_n; \mathbb{Z}_p)$$

was computed by Mikael Vejdemo Johansson in his 2008 thesis. Mikael will arrive in Millersville on March 25 and be in residence for approximately one month. This discussion will continue in his colloquium talk on March 27.