Tensor Products of A$_{(\infty)}$-algebras and Group Cohomology

Presented by

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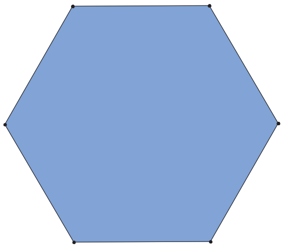
The Permutahedron P_n

- $P_n = \text{convex hull}\,\{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \in S_n\}$
- Vertices $v_1, \ldots, v_{n!}$ in the hyperplane $x_1 + \cdots + x_n = \binom{n}{2}$

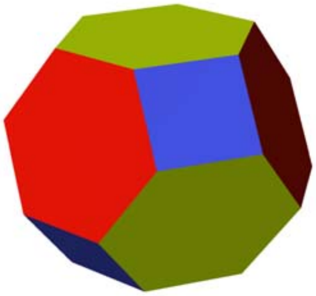
P_1 is a point $*$

P_2 is a closed interval I

P_3 is a plane hexagonal region



P_4 is a solid truncated octahedron



P_n is an $(n-1)$-dim'l convex polyhedron

Combinatorics of P_n

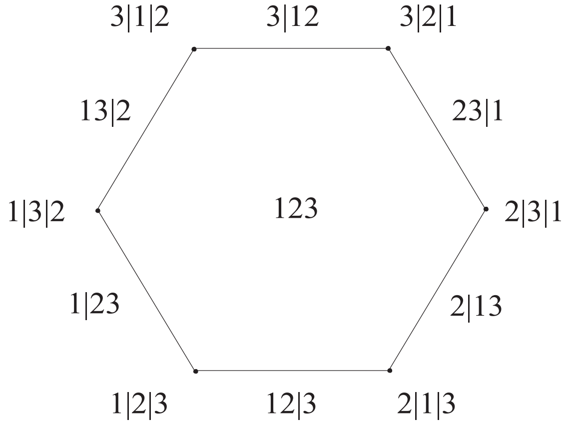
Let $\underline{n} = \{1, 2, \ldots n\}$

- $\{\text{Faces in codim}\, p\} \leftrightarrow \{\text{Partitions } U_1 | \cdots | U_{p+1} \text{ of } \underline{n}\}$

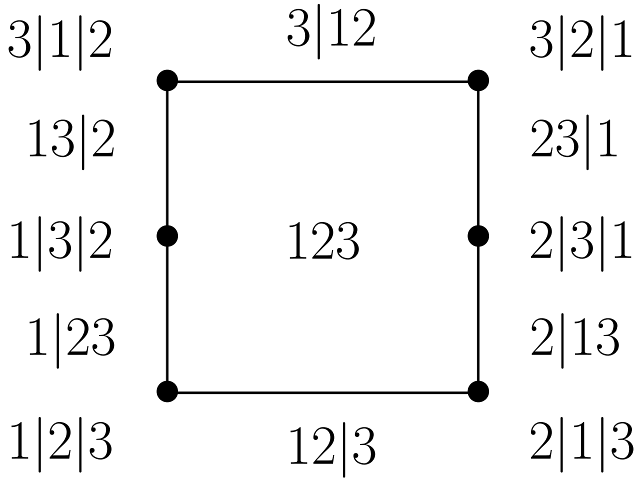
$P_1 :\ * \leftrightarrow \{1\}$

$P_2 :$ edge $\leftrightarrow \{12\}$; vertices $\leftrightarrow \{1|2, 2|1\}$

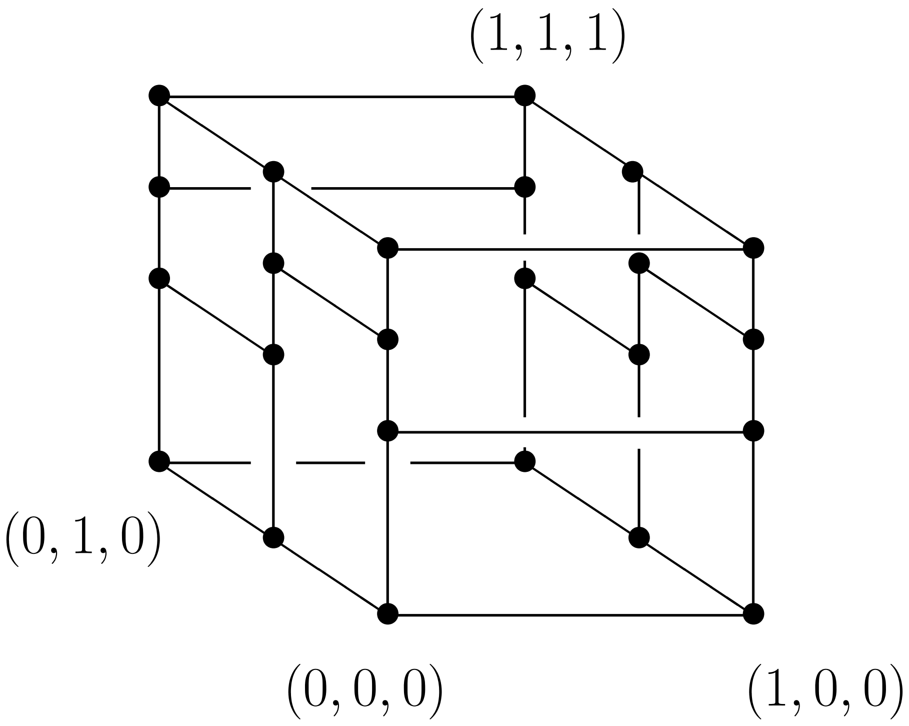
$P_3 :$



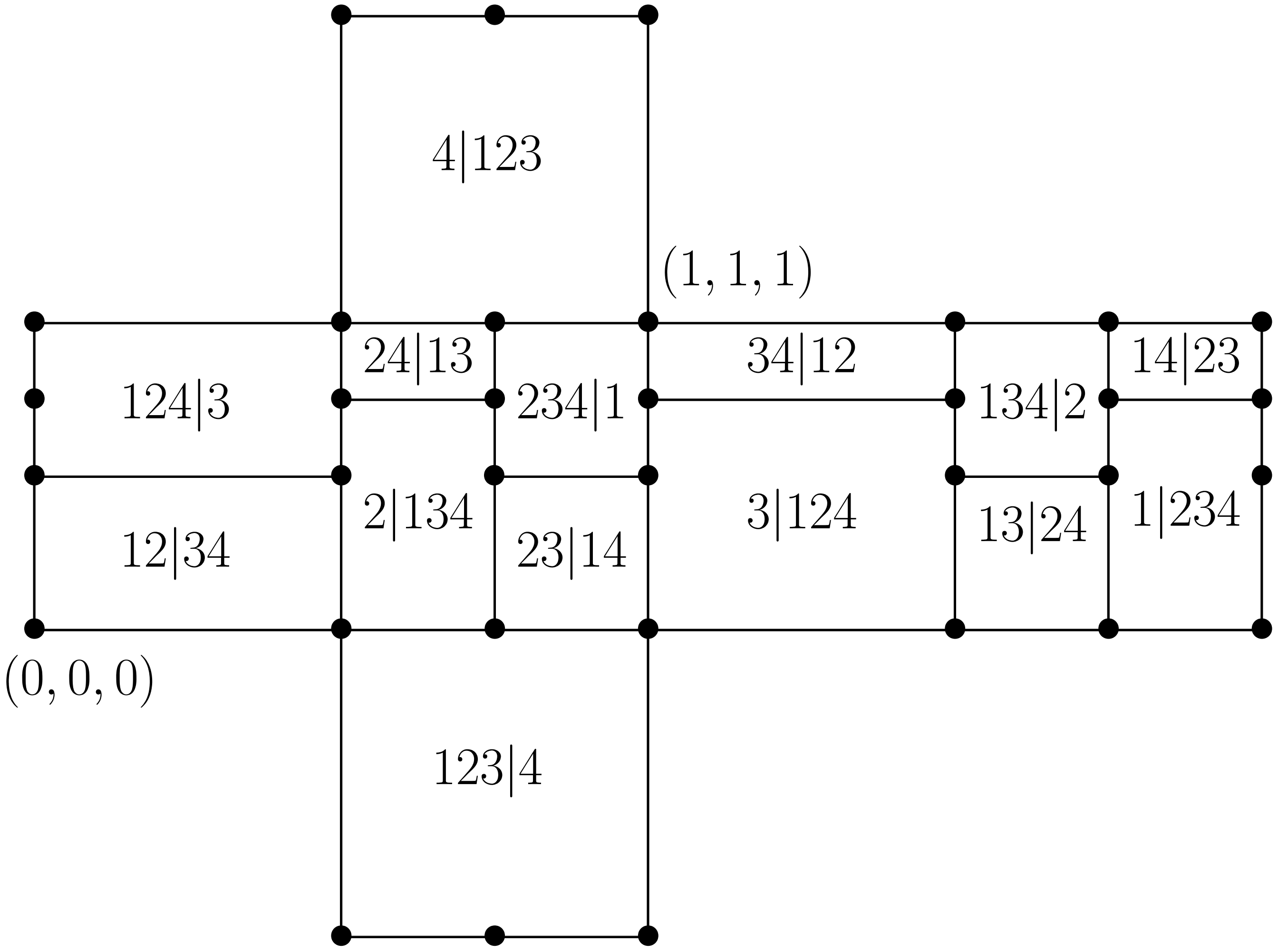
as a subdivision of $P_2 \times I$:



P_4 as a subdivision of $P_3 \times I$:



The 2-faces of P_4:



Inductively, P_{n+1} can be obtained as a subdivision of $P_n \times I$

Cellular Chains

Fix a ground field F; let X be a cellular complex

- *Cellular k-chains of X over F* are elements of

  the F-vector space $C_k(X) = \langle k\text{-cells of } X\rangle$

- The *graded vector space of cellular chains of X* is

$$C_*(X) = \bigoplus_{k\geq 0} C_k(X)$$

- *Cellular boundary* ∂ is defined on the k-cells of X

  and extends to a linear map

$$\partial : C_*(X) \to C_{*-1}(X)$$

  that satisfies $\partial \circ \partial = 0$

- A *differential* on a graded vector space V_* is a

  linear map $d : V_* \to V_{*-1}$ such that $d \circ d = 0$

- (V_*, d) is a *__D__ifferential __G__raded __V__ector __S__pace* (DGVS)

- ∂ is a differential on $C_*(X)$

- $(C_*(X), \partial)$ is DGVS

- Let (V_*, d_V) and (W_*, d_W) be DGVS

  A linear map $f : V_* \to W_*$ is a *chain map* if

$$d_W \circ f = f \circ d_V$$

- A map $f : X \to Y$ of cellular complexes is *cellular*

  if f sends each cell of X to a cell of Y

- Every map $f : X \to Y$ of cellular complexes is

  homotopic to a cellular map $g : X \to Y$

- A cellular map $g : X \to Y$ induces a chain map

  $g : C_*(X) \to C_*(Y)$ by restricting to the

  cells of X and extending linearly

Diagonal Approximations

- The *geometric diagonal* $\Delta : X \to X \times X$ is

  defined by $\Delta(x) = x \times x$

- A *diagonal approximation* of Δ is a cellular map

  $\Delta_X : X \to X \times X$ homotopic to Δ

- Faces of the n-simplex $s^n = [012 \cdots n]$ are

  indexed by all (increasing) subsequences

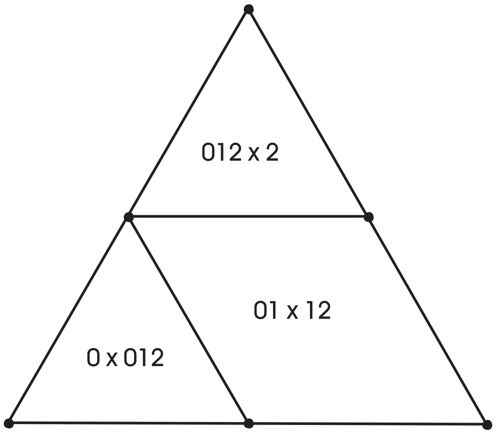
  $$[a_0 \cdots a_k] \subseteq [012 \cdots n]$$

- $\partial([a_1 \cdots a_k]) = \sum (-1)^i [a_0 \cdots \widehat{a_i} \cdots a_k]$

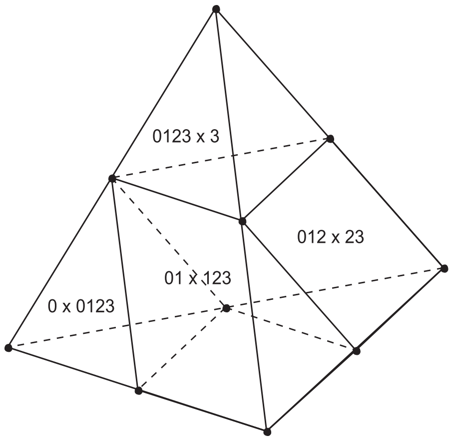
- Alexander-Whitney diagonal approximation:

  $$\Delta_{s^n}([012 \cdots n]) = \sum_{i=0}^{n} [01 \cdots i] \times [i \cdots n]$$

- A diagonal approximation Δ_X determines a tessellation of X transforming Δ_X into an inclusion



The tessellation of s^2 determined by A-W



The tessellation of s^3 determined by A-W

- The free linear extension of ∂ to $C_*(X) \otimes C_*(X)$

  is a differential $\partial \otimes 1 + 1 \otimes \partial$

- A *diagonal* on $C_*(X)$ is a chain map

$$\Delta_X : C_*(X) \to C_*(X \times X) \approx C_*(X) \otimes C_*(X)$$

$$\Delta_X \circ \partial = (\partial \otimes 1 + 1 \otimes \partial) \circ \Delta_X$$

- A-W is strictly coassociative and homotopy

  cocommutative on cellular chains

The Saneblidze-Umble Diagonal Δ_P

- A matrix E is a *step matrix* if for some n

  1. Each element of $\underline{n}$ appears as an entry of E exactly once
  2. The elements of $\underline{n}$ in each row and column of E form an increasing contiguous block
  3. Each diagonal parallel to the main diagonal of E contains exactly one element of $\underline{n}$

- A typical step matrix with $n = 9$:

$$E = \begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline & & & 5 \\ \hline & & 4 & 6 \\ \hline 1 & 3 & 8 & \\ \hline 7 & & & \\ \hline 9 & & & \\ \hline \end{array}$$

- Right-shift operator on $G^{q\times p} = (g_{i,j})$

  Let $M_j \subset \{g_{*,j}\}$

  If $M_j \neq \varnothing$, then $\min M_j = g_{k,j}$ for some k

  If $\min M_j > \max\{g_{*,j+1}\}$ and

  $g_{t,j+1} = 0$ for $k \leq t \leq q$, obtain $R_{M_j}G$

  by interchanging $g_{i,j} \in M_j$ and $g_{i,j+1}$

  Otherwise, define $R_{M_j}G = G$

- Down-shift operator on $G^{q\times p} = (g_{i,j})$

  Let $N_i \subset \{g_{i,*}\}$

  If $N_i = \varnothing$, then $\min N_i = g_{i,k}$ for some k

  If $\min N_i > \max\{g_{i+1,*}\}$ and

  $g_{i+1,t} = 0$ for $k \leq t \leq p$, obtain $D_{N_i}G$

  by interchanging $g_{i,j} \in N_i$ and $g_{i+1,j}$

  Otherwise, define $D_{N_i}G = G$.

- $F = D_{N_p} D_{N_{p-1}} \cdots D_{N_1} R_{M_q} R_{M_{q-1}} \cdots R_{M_1} E$

  is a *derived matrix* iff E is a step matrix

- Step matrices are derived matrices via $M_i = N_j = \varnothing$

$$E = \begin{array}{|c|c|c|}\hline & 2 & 3 \\ \hline 1 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad \longrightarrow \quad D_{\varnothing} D_{\varnothing} R_5 R_{\varnothing} E = \begin{array}{|c|c|c|}\hline & 2 & 3 \\ \hline 1 & & 5 \\ \hline 4 & & \\ \hline \end{array}$$

- $A_1 | A_2 | \cdots | A_p \times B_q | B_{q-1} | \cdots | B_1$ is a

  Complementary Pair (CP) of partitions iff B_i

  and A_j are the rows and columns of a $q \times p$

  derived matrix

  Above: $E \leftrightarrow 14|25|3 \times 4|15|23$

- $\{$Derived matrices$\} \leftrightarrow \{$CPs$\}$
- $\{$CPs$\} \leftrightarrow \{$Faces of $P_n \times P_n\}$

- **The S-U diagonal** Δ_P (2004)

Let e^{n-1} be the top dim'l face of P_n; define

$$\Delta_P(e^0) = e^0 \otimes e^0$$

Having defined Δ_P on $C_*(P_k)$, $k \leq n-1$,

define Δ_P on $C_{n-1}(P_n)$ by

$$\Delta_P(e^{n-1}) = \sum_{\substack{(p,q)\text{-CPs } u\times v \\ p+q=n+1}} \pm\, u \otimes v$$

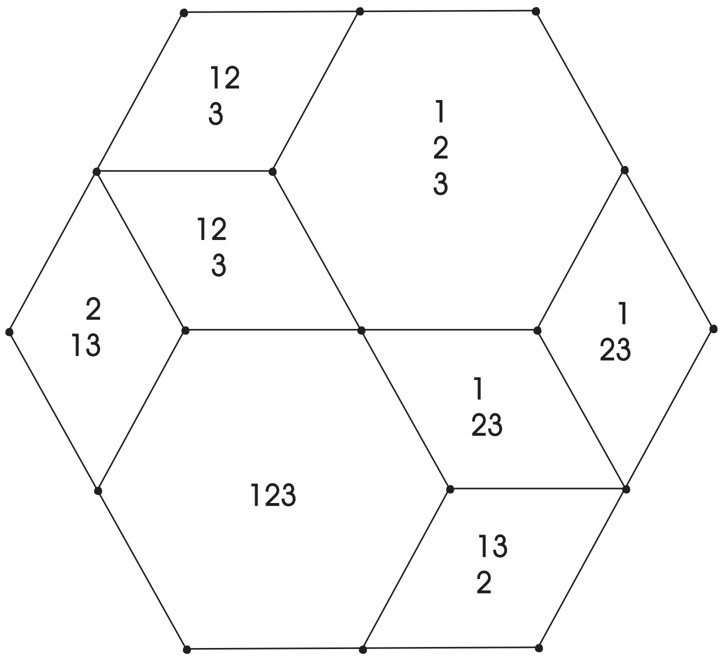
and extend multiplicatively to $C_*(P_n)$

- When $n = 3$, there are 8 derived matrices

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline & 2 \\ \hline 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline & 1 \\ \hline 2 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \xrightarrow{R} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \xrightarrow{D} \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & 3 \\ \hline \end{array}$$

$\Delta_P(P_3)$:



The Associahedron K_n (Stasheff 1963)

- $K_n = P_{n-1}/\sim$

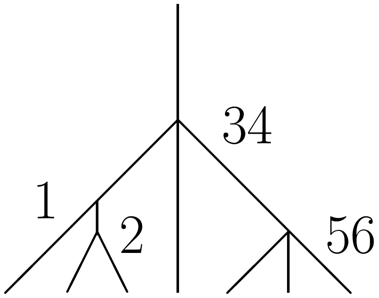
- $\{$Codim p faces of $P_{n-1}\} \leftrightarrow$

  $\{$PLTs with n leaves and $p+1$ levels$\}$

  (PLT = <u>P</u>lanar rooted <u>L</u>eveled <u>T</u>ree)

- $\{$PLTs with n leaves and $p+1$ levels$\} \leftrightarrow$

  $\{$Partitions of $\underline{n-1}$ of length $p+1\}$



$\leftrightarrow \quad 256|1|34$

Given a PLT T, number the leaves from left to right and assign the label i to the node at which the branch of leaf i meets the branch of leaf $i+1$. Let $U_j =$ $\{$labels of nodes in level $j\}$; then $T \leftrightarrow U_1|\cdots|U_{p+1}$

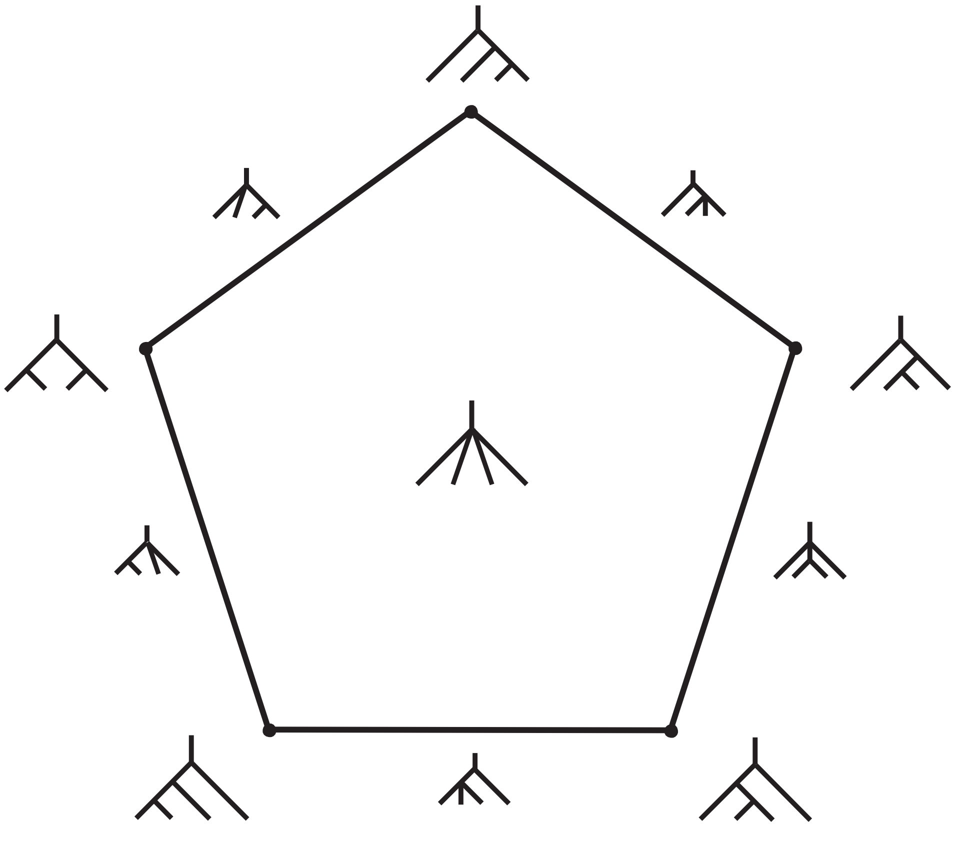
- $\{\text{Faces of } K_n\} \leftrightarrow \{\text{PRTs with } n \text{ leaves}\}$

(PRT = Planar Rooted Tree)

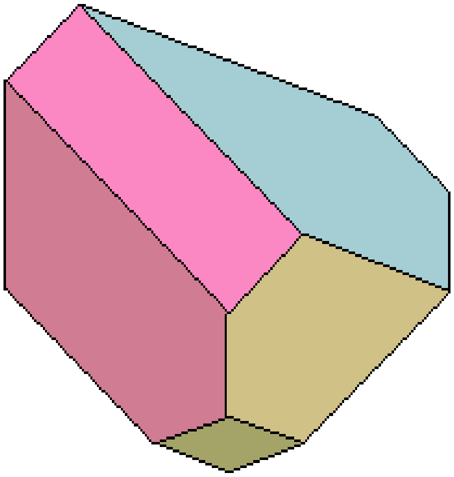
$K_2 = * \leftrightarrow \{$  $\}$

$K_3 = I \leftrightarrow \{$  $\}$; vertices $\leftrightarrow \{$  ,  $\}$

K_4 is a plane pentagonal region



K_5:



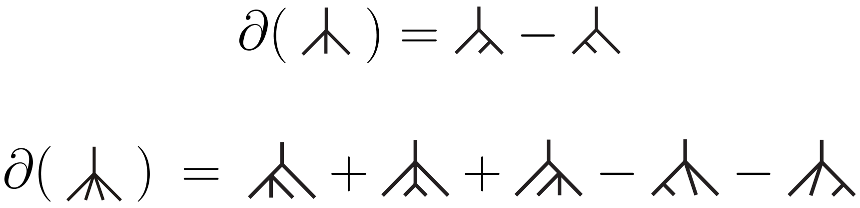
- $\{$PRTs with n leaves$\} \leftrightarrow$

  $\{$Parenthesizations of n variables$\}$

- $\partial : C_* (K_n) \to C_{*-1} (K_n)$ is defined by

  $$\partial (T) = \sum \pm T_i,$$

  where T_i has one more node than T and the T_i's range over all possible trees obtained from T by grafting in a branch at some node of valance > 3



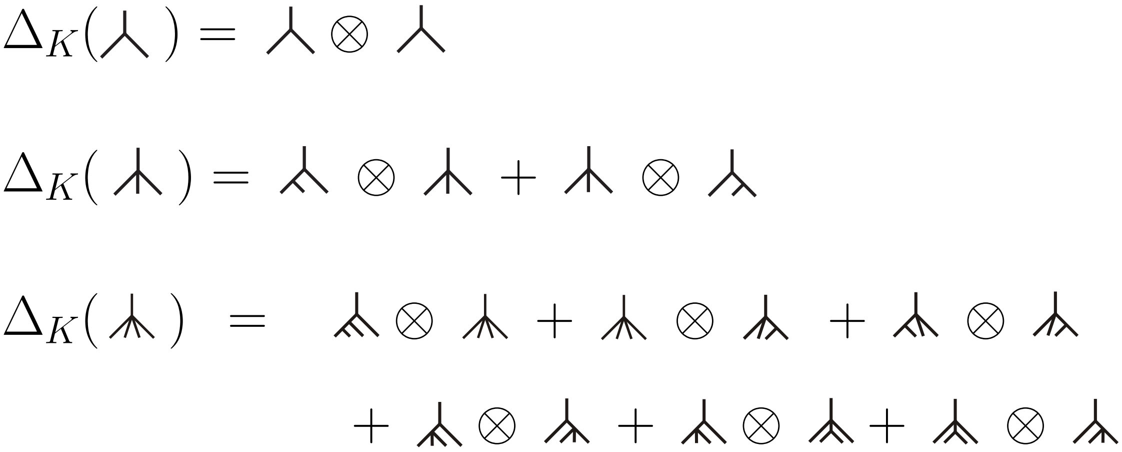
- Forgetting levels defines Tonks' projection (1997)

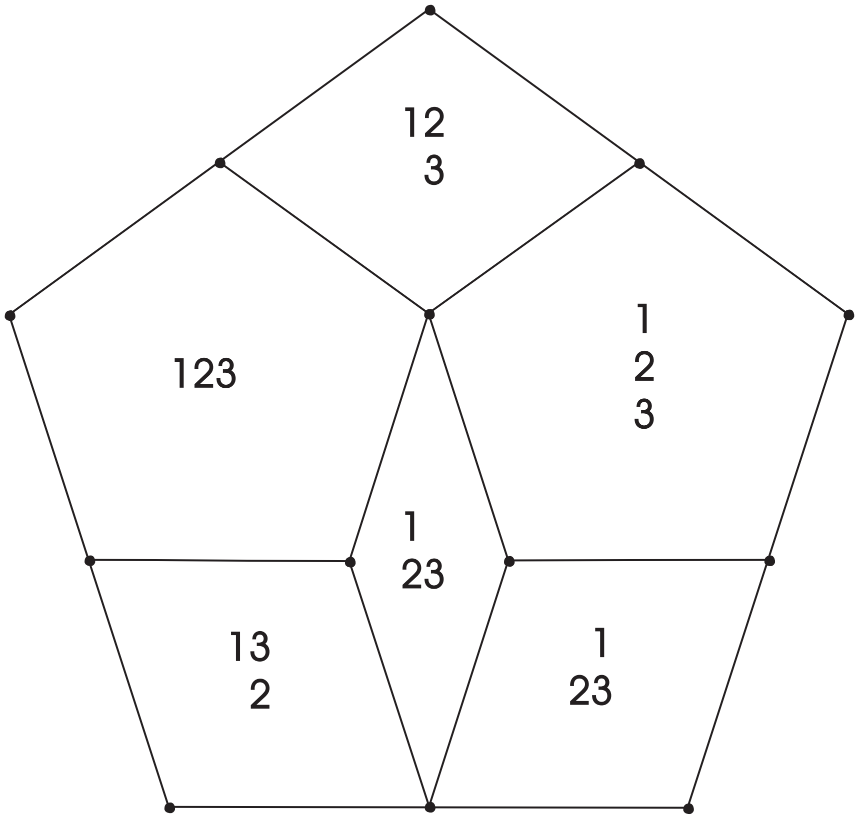
$$\theta : P_{n-1} \longrightarrow K_n$$

- Faces of P_{n-1} indexed by PLTs with multiple nodes in the same level degenerate under θ; corresponding generators lie in the kernel of the induced map $\theta : C_*(P_{n-1}) \to C_*(K_n)$

- The S-U diagonal Δ_K is defined by

$$\Delta_K \theta = (\theta \otimes \theta)\Delta_P$$


Δ_K
Δ_K
Δ_K


12
3
123
1
2
3
1
23
13
2
1
23

A_∞**-algebras** (Stasheff 1963)

- Given a DGVS (A, d) consider $\{Hom^p(A^{\otimes n}, A)\}_{p\in\mathbb{Z}}$

- Free linear extension of d to $A^{\otimes n}$ is a differential

$$d_n \,=\, d \otimes 1^{\otimes n-1} + 1 \otimes d \otimes 1^{n-2} + \cdots + 1^{\otimes n-1} \otimes d$$

- d induces a differential δ on $Hom^*(A^{\otimes n}, A)$ via

$$\delta(f) \,=\, d \circ f - (-1)^p f \circ d_n$$

- Thus $(Hom^*(A^{\otimes n}, A), \delta)$ is a DGVS

- Identify top dimensional cell e^{n-2} of K_n with a

  multilinear operation $\mu_n \in Hom^{n-2}(A^{\otimes n}, A)$

- An A_∞-*algebra* is a DGVS (A, d) together

  with a family of operations

  $$\left\{\mu_n \in Hom^{2-n}\left(A^{\otimes n}, A\right)\right\}_{n\geq 2}$$

  and a family of chain maps

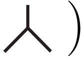
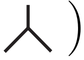
  $$\left\{\zeta : C_*\left(K_n\right) \to Hom^{2-*}\left(A^{\otimes n}, A\right)\right\}_{n\geq 2}$$

  such that $\zeta\left(e^{n-2}\right) = \mu_n$ and $\zeta\left(T\right)$ on a proper face

  $T \subset K_n$ is the composition of μ_k's specified by T

- **Structure Relations**

1. $d\mu_2 - \mu_2\left(d \otimes 1 + 1 \otimes d\right) = \delta\mu_2$

   $= \delta\zeta($  $) = \zeta\partial($  $) = 0$

   $\Rightarrow \mu_1$ is a derivation of μ_2

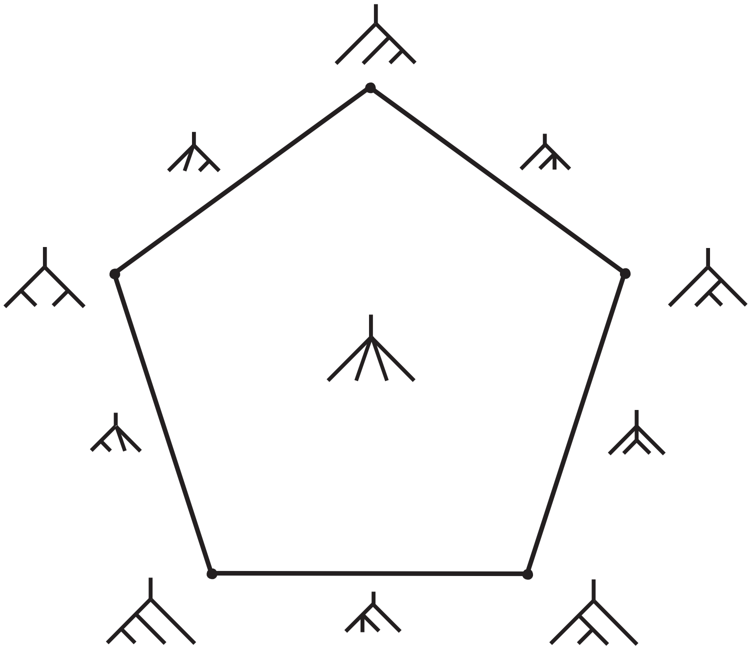
2. $d\mu_3 + \mu_3 d_3 = \delta\mu_3$

   $= \delta\zeta($  $) = \zeta\partial($  $) = \zeta($  $-$  $)$

   $= \mu_2\left(1 \otimes \mu_2\right) - \mu_2\left(\mu_2 \otimes 1\right)$

   $\Rightarrow \mu_2$ is homotopy associative via μ_3

3. Stasheff's Pentagon Relation:



$d\mu_4 - \mu_4 d_4 = \delta\mu_4 = \delta\zeta($  $) = \zeta\partial($  $)$

$= \zeta($  $+$  $+$  $-$  $-$  $)$

$$= \mu_2\,(\mu_3 \otimes 1) + \mu_3\,(1 \otimes \mu_2 \otimes 1) + \mu_2\,(1 \otimes \mu_3)$$

$$-\mu_3\,(\mu_2 \otimes 1 \otimes 1) - \mu_3\,(1 \otimes 1 \otimes \mu_2)$$

Tensor Products of A_∞-algebras (S-U 2000)

Let (A, ζ_A) and (B, ζ_B) be A_∞-algebras. The *induced A_∞-algebra structure* $\zeta_{A\otimes B}$ is given by the composition

$$\begin{tikzcd}
C_*(K_n) \arrow[r, "\zeta_{A\otimes B}"] \arrow[d, "\Delta_K"'] & Hom((A\otimes B)^{\otimes n}, A\otimes B) \\
C_*(K_n)\otimes C_*(K_n) \arrow[r, "\zeta_A\otimes\zeta_B"'] & Hom(A^{\otimes n}\otimes B^{\otimes n}, A\otimes B) \arrow[u, "(\sigma_{2,n})^*"']
\end{tikzcd}$$

where $\sigma_{2,n} : (A\otimes B)^{\otimes n} \to A^{\otimes n}\otimes B^{\otimes n}$ is the canonical permutation of tensor factors:

$$\sigma_{2,3} : a_1|b_1 \otimes a_2|b_2 \otimes a_3|b_3 \mapsto a_1|a_2|a_3 \otimes b_1|b_2|b_3$$

- The A_∞-algebra operations $\{\Phi_n\}$ on $A\otimes B$ are

$$\Phi_n \;=\; \zeta_{A\otimes B}\left(e^{n-2}\right) = \left[(\zeta_A\otimes\zeta_B)\,\Delta_K\left(e^{n-2}\right)\right]\sigma_{2,n}$$

and in particular

$\Phi_2 = \big[(\zeta_A\otimes\zeta_B)\,($  $\otimes$  $)\big]\,\sigma_{2,2}$

$\Phi_3 = \big[(\zeta_A\otimes\zeta_B)\,($  $\otimes$  $+$  $\otimes$  $)\big]\,\sigma_{2,3}$

$\Phi_4 = \big[(\zeta_A\otimes\zeta_B)\,($  $\otimes$  $+$  $\otimes$ 

$+$  $\otimes$  $+$  $\otimes$  $+$  $\otimes$ 

$+$  $\otimes$  $)\big]\,\sigma_{2,4}$

$\vdots \qquad\qquad\qquad\qquad \vdots$

Tensor Products of Simple A_∞-algebras

- A *simple A_∞-algebra* has exactly one non-trivial operation μ_n for $n \geq 3$

- **Example:** $A = \Lambda(x, y)$ with $|x| = 1$, $|y| = 2$, and $d = 0$. Let $\mu = \mu_2$ denote the product and define

$$\mu_3\left(x^i y^p | x^j y^q | x^k y^r\right) = \begin{cases} y^{p+q+r+1}, & ijk = 1,\ pqr \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

One checks that the two non-trivial structure relations hold:

1. $\mu(\mu_3 \otimes 1) + \mu_3(1 \otimes \mu \otimes 1) + \mu(1 \otimes \mu_3)$
$$= \mu_3(\mu \otimes 1 \otimes 1) + \mu_3(1 \otimes 1 \otimes \mu)$$

2. $\mu_3(\mu_3 \otimes 1 \otimes 1 + 1 \otimes \mu_3 \otimes 1 + 1 \otimes 1 \otimes \mu_3) = 0$

Conclude that (A, μ, μ_3) is a simple A_∞-algebra

Over $\mathbb{Z}_3$ there are exactly two non-trivial higher order operations Φ_3 and Φ_4 on $A \otimes A$:

$$\Phi_3 = [\mu(\mu \otimes 1) \otimes \mu_3 + \mu_3 \otimes \mu(1 \otimes \mu)]\,\sigma_{2,3}$$

$$\begin{aligned}\Phi_4 = [&\mu(\mu_3 \otimes 1) \otimes \mu_3(1 \otimes \mu \otimes 1) \\ &+\mu(\mu_3 \otimes 1) \otimes \mu(1 \otimes \mu_3) \\ &+\mu_3(1 \otimes \mu \otimes 1) \otimes \mu(1 \otimes \mu_3) \\ &+\mu_3(\mu \otimes 1 \otimes 1) \otimes \mu_3(1 \otimes 1 \otimes \mu)]\sigma_{2,4}\end{aligned}$$

Proposition: Let X and Y be topological spaces; let F be a field. Then

$$H_*(X \times X; F) \approx H_*(X; F) \otimes H_*(X; F)$$

is an A_∞-algebra with the induced A_∞-structure.

Application to Group Cohomology

Theorem (Madson 02): $H^*(C_n; \mathbb{Z}_p)$ is a simple A_∞-algebra. As a DGA, $H^*(C_n; \mathbb{Z}_p) = \Lambda(x, y)$ with $|x| = 1$, $|y| = 2$, and $d = 0$. In the A_∞-structure μ_2 is the cup product and

$$\mu_n(xy^{p_1}|\cdots|xy^{p_n}) = y^{1+\boldsymbol{\Sigma} p_i}$$

Theorem (Vejdemo-Johansson 08): If $m, n \geq 4$, the A_∞-algebra structure on $H^*(C_m \times C_n; \mathbb{Z}_p)$ has non-trivial operations of arities

$$k(m-2) + k(n-2) + 2,$$

$$(k-1)(m-2) + k(n-2) + 2,$$

$$k(m-2) + (k-1)(n-2) + 2.$$

Theorem (U): The induced A_∞-structure on $H^*(C_3 \times C_3; \mathbb{Z}_3)$ has exactly two non-trivial operations of arities 3 and 4.