Transferring $A_\infty$-Structures from Chains to Homology

Joint work with Samson Saneblidze

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4D Digital Imaging Seminar

24 September 2009
Goal of the Talk

To understand the following statement:

**Theorem.** *If* $A$ *is an* $A_\infty$-*structure over a field* $k$, *there is an induced* $A_\infty$-*structure on* $H(A; k)$.
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- Many have studied the transfer of $A_\infty$-algebra structure, including Kadeishvili, Huebschmann, Kontsevich, Soibelman, Merkulov, Markl, Real, M-J Jiménez, Berciano, to name a few...
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- Saneblidze observed that the "Transfer Problem" is simpler at the level of hom groups

- Our method relaxes the conditions under which the transfer of $A_\infty$-algebra structure occurs, and transfers $A_\infty$-bialgebra structure as well
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$\dim (K_n) = n - 2$
Background: A-infinity Algebras

- Cellular chains on associahedra $K = \{K_n\}_{n \geq 2}$ realize the operad $\mathcal{A}_\infty$

- $\dim (K_n) = n - 2$

- An $A_\infty$-algebra is a DGM $(A, d)$ together with a family of operations

$$\{\mu^n \in \text{Hom}^{n-2}(A^{\otimes n}, A)\}_{n \geq 2}$$

and a map $\varphi : \mathcal{A}_\infty \rightarrow \{\text{Hom}(A^{\otimes n}, A)\}_{n \geq 1}$ of non-$\Sigma$ operads
Background: A-infinity Bialgebras

- Cellular chains on matrashedra $KK = \{KK_{n,m}\}$ realize the matrad $\mathcal{H}_\infty$
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- $\dim (KK_{n,m}) = m + n - 3$

- An $A_\infty$-bialgebra is DGM $(H, d)$ together with a family of operations

  \[ \{ \omega_m^n \in \text{Hom}^{m+n-3} (H^\otimes m, H^\otimes n) \}_{m,n \geq 1; m+n \geq 3} \]

  and a map $\varphi : \mathcal{H}_\infty \rightarrow \text{End} (TH)$ of matrads
Given DGMs \((A, d_A)\) and \((B, d_B)\), define \(\nabla\) on \(\text{Hom}(B, A)\) by

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\nabla(u) = d_A u - (-1)^{|u|} ud_B
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A chain map \(g : B \to A\) induces a cochain map
\[
\tilde{g} : (\text{Hom}(B^\otimes n, B) ; \nabla_B) \to (\text{Hom}(B^\otimes n, A) ; \nabla)
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via \(g(u) = gu\)
Introduction and Overview

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- **Theorem 1.** Let \(A\) be an \(A_\infty\)-algebra, \(B\) be a DGM, and \(g : B \to A\) be a chain map. If \(\bar{g}\) is a quasi-isomorphism, then \(g\) induces an \(A_\infty\)-algebra structure on \(B\).
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If \(\tilde{g}\) is a quasi-isomorphism, so is \(g\), but not conversely.
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- If \(\bar{g}\) is a quasi-isomorphism, so is \(g\), but not conversely

- However, the converse holds whenever \(B\) is free
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Markl assumes $g$ has a right-homotopy inverse $f$

$$1 - gf = d_A \phi + \phi d_A$$
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• All assume there is a homotopy operator $\phi : A \to A$
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Our algorithm requires neither freeness nor a homotopy operator
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Our algorithm requires neither freeness nor a homotopy operator

$B = H(A)$ is an important special case of interest in this talk
**Corollary 1.** Let $A$ be an $A_\infty$-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^k \text{Hom}(H^\otimes k, X) = 0$ for $k \geq 2$, there is an induced $A_\infty$-algebra structure on $H$. 

**Example 1.** We exhibit a DGA $A$ with an induced non-trivial $A_\infty$-algebra structure on $H(A)$ that cannot be computed using standard techniques.

**Corollary 2.** Given an $A_\infty$-bialgebra $A$ over a field $k$, there is an induced $A_\infty$-bialgebra structure on $H$. 

**Example 2.** Let $X$ be a space. There is an induced $A_\infty$-bialgebra structure on $H(\Omega X; k)$. 

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Key Points in the Talk

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Key Points in the Talk

- **Corollary 1.** Let $A$ be an $A_\infty$-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* \text{Hom}(H \otimes^k, X) = 0$ for $k \geq 2$, there is an induced $A_\infty$-algebra structure on $H$.

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- **Example 2.** Let $X$ be a space. There is an induced $A_\infty$-bialgebra structure on $H_*(\Omega X; k)$.
Given $A_{\infty}$-algebras $A$ and $B$, and a chain map $g : B \to A$, a morphism

$$G = g + g_2 + g_3 + \cdots : B \Rightarrow A$$

is defined in terms of parameter spaces $\{J_n\}_{n \geq 1}$, called multiplihedra.
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- $J_1 = \ast$ is identified with the cochain $g \in \text{Hom}^0 (B, A)$
- $J_2 = I$ is identified with the cochain $g_2 \in \text{Hom}^1 (B \otimes B, A)$
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- $J_2 = I$ is identified with the cochain $g_2 \in Hom^1 (B \otimes B, A)$

- The endpoints of $J_2$ are identified with components of the coboundary

$$\nabla g_2 = \mu_A (g \otimes g) - g \mu_B$$
$J_3$ is an hexagonal plane region identified with $g_3 \in Hom^2 (B \otimes^3, A)$

\[ g \mu^3_B \]

\[ g_2 (\mu_B \otimes 1) \]
\[ g_2 (1 \otimes \mu_B) \]
\[ \mu_A (g_2 \otimes g) \]
\[ \mu_A (g \otimes g_2) \]
\[ \mu^3_A (g \otimes g \otimes g) \]

\[ \nabla g_3 = \mu^3_A g \otimes^3 + \mu_A (g_2 \otimes g - g \otimes g_2) + g_2 (\mu_B \otimes 1 - 1 \otimes \mu_B) - g \mu^3_B \]
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A–infinity Maps and Multiplihedra

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- \(\mu^n_B\) appears in exactly one component of \(\nabla g_n\), namely \(g \mu^n_B\).
- Let \(Q^{n-2}\) denote the cell of \(J_n\) identified with the cochain \(g \mu^n_B\).
- Let \(\Theta_n\) denote the cochain identified with \(\partial J_n - \text{int} Q^{n-2}\).
- Then \(\partial Q^{n-2}\) is identified with the coboundary \(\nabla \Theta_n\).
Transfer Problem 1: Let $A$ be an $A_{\infty}$-algebra, let $B$ be a DGM, and let $g : B \to A$ be a chain map. Given $\{\mu^i_B, g_i\}_{2 \leq i \leq n}$ construct $\mu^n_B$ and $g_n$ so that

$$\nabla g_n = \Theta_n - g \mu^n_B$$
**First Transfer Theorem**

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- Transfer Problem 1 has a solution whenever $\bar{g}$ is a quasi-isomorphism.
First Transfer Theorem

- **Transfer Problem 1:** Let $A$ be an $A_\infty$-algebra, let $B$ be a DGM, and let $g : B \to A$ be a chain map. Given $\{\mu_B^i, g_i\}_{2 \leq i < n}$ construct $\mu^n_B$ and $g_n$ so that

$$\nabla g_n = \Theta_n - g \mu^n_B$$

- Transfer Problem 1 has a solution whenever $\bar{g}$ is a quasi-isomorphism

- **Theorem 1.** If $\bar{g}$ is a quasi-isomorphism, then

  (i) $g$ transfers the $A_\infty$-algebra structure from $A$ to $B$; the induced structure on $B$ is unique up to automorphism.

  (ii) $g$ extends to a map $G : B \Rightarrow A$ of $A_\infty$-algebras.
**Proposition 1.** Let $A$ be an $A_\infty$-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* \text{Hom}(H^{\otimes k}, X) = 0$ for $k \geq 2$, there is a cycle-selecting homomorphism $g : H \to A$ such that $\bar{g}$ is a quasi-isomorphism.
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**Corollary 1.** Let $A$ be an $A_\infty$-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* \text{Hom} \left( H^k, X \right) = 0$ for $k \geq 2$, there is an induced $A_\infty$-algebra structure on $H$. 
Let \((A, d, \mu^n)_{n \geq 2}\) be an \(A_\infty\)-algebra, \(g : H \to A\) a cycle-selecting hom, and assume that an \(A_\infty\)-structure \((H, \mu^n_H)_{n \geq 2}\) has been constructed.

- Thinking of \(g \in \text{Hom}(H, A)\), note that \(\nabla g = dg = 0\)
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- \(\otimes_2 = \mu(g \otimes g) \in \text{Hom}^0(H \otimes H, A)\)

- \(\nabla \otimes_2 = d\mu(g \otimes g) = \mu(dg \otimes g + g \otimes dg) = 0\)
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Since $\partial (\partial J_n - \text{int} Q^{n-2}) = \partial Q^{n-2}$ we have

$\nabla \exists_n = \nabla g\mu^n_H = dg\mu^n_H = 0$

Therefore $\exists_n$ is a cocycle for all $n \geq 2$
Computing the Transfer: The Fundamental Cocycle

- For \( n \geq 3 \), recall that \( g\mu_H^n \) is identified with \( Q^{n-2} \)
- \( \Theta_n \in \text{Hom}^{n-2}(H^{\otimes n}, A) \) is identified with \( \partial J_n - \text{int} \ Q^{n-2} \)
- Since \( \partial (\partial J_n - \text{int} \ Q^{n-2}) = \partial Q^{n-2} \) we have
- \( \nabla \Theta_n = \nabla g\mu_H^n = d\mu^n_H = 0 \)
- Therefore \( \Theta_n \) is a cocycle for all \( n \geq 2 \)
- We refer to \( \Theta_n \) as the fundamental \( n \)-cocycle
Now suppose $\tilde{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu^i_H, g_i\}_{2 \leq i < n}$ has been constructed

Solve the linear system $dx = n \mu^i_H g_i$

Choose a particular solution $g_n$

Then $g + g_2 + \ldots + g_n$ is an $A_\infty$-map
Now suppose $\tilde{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu^i_H, g_i\}_{2 \leq i < n}$ has been constructed.

- Since $\nabla \Theta_n = 0$, we may define $\mu^n_H = (\tilde{g}_*)^{-1} [\Theta_n]$.
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- Since $\nabla \Theta_n = 0$, we may define $\mu^n_H = (\tilde{g}_*)^{-1} [\Theta_n]$.

- Then $g \mu^n_H \in [\Theta_n]$ and $\Theta_n - g \mu^n_H$ is a coboundary.
Computing the Transfer: The Induction

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- Solve the linear system $dx = \Theta_n - g\mu^n_H$
Computing the Transfer: The Induction

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- Then \( g\mu^n_H \in [\Theta_n] \) and \( \Theta_n - g\mu^n_H \) is a coboundary

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- Choose a particular solution \( g_n \)
Computing the Transfer: The Induction

Now suppose $\tilde{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu_{\tilde{g}}^i, g_i\}_{2 \leq i < n}$ has been constructed.

- Since $\nabla \Theta_n = 0$, we may define $\mu^n_H = (\tilde{g}_*)^{-1} [\Theta_n]$. 

- Then $g\mu^n_H \in [\Theta_n]$ and $\Theta_n - g\mu^n_H$ is a coboundary.

- Solve the linear system $dx = \Theta_n - g\mu^n_H$.

- Choose a particular solution $g_n$.

- Then $g + g_2 + \cdots + g_n$ is an $A_n$-map.
Example 1. Consider the DGM

\[
\begin{array}{ccccccccc}
M^0 & \to & 0 & \to & M^2 & \to & M^3 & \to & M^4 & \to & 0 & \to & \cdots \\
Z & \to & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \to & \mathbb{Z}_4 & \to & \mathbb{Z}_2 \\
(a_2, b_2) & \mapsto & (0, 2a_3) & & a_3 & \mapsto & a_4
\end{array}
\]
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(a_2, b_2) & (0, 2a_3) & a_3 & a_4
\end{array}
\]

\[ A = T^a M / (a_2^2 + a_4, a_3^2, a_4a_3 + a_3a_4, (a_2a_3 + a_3a_2)^2, a_ib_2, b_2a_i, b_2^2) \]
**Example 1.** Consider the DGM

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\[ A = T^a M / (a_2^2 + a_4, a_3^2, a_4a_3 + a_3a_4, (a_2a_3 + a_3a_2)^2, a_ib_2, b_2a_i, b_2^2) \]

\[ A \text{ has no Hodge decomposition since } \mathbb{Z}_4 \text{ contains a non-cycle } a_3 \text{ and a boundary } 2a_3. \text{ But } \mathbb{Z}_4 \text{ does not split as } \mathbb{Z}_2 \oplus \mathbb{Z}_2. \]
**Example 1.** Consider the DGM

\[ M^0 \rightarrow 0 \rightarrow M^2 \rightarrow M^3 \rightarrow M^4 \rightarrow 0 \rightarrow \cdots \]

\[ \mathbb{Z} \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \]

\[ (a_2, b_2) \mapsto (0, 2a_3) \]

\[ a_3 \mapsto a_4 \]

\[ A = \frac{T^a M}{(a_2^2 + a_4, a_3^2, a_4 a_3 + a_3 a_4, (a_2 a_3 + a_3 a_2)^2, a_i b_2, b_2 a_i, b_2^2)} \]

\[ A \text{ has no Hodge decomposition since } \mathbb{Z}_4 \text{ contains a non-cycle } a_3 \text{ and a boundary } 2a_3. \text{ But } \mathbb{Z}_4 \text{ does not split as } \mathbb{Z}_2 \oplus \mathbb{Z}_2. \]

\[ H^n (A) = \begin{cases} 
\mathbb{Z} & n = 0, \\
\mathbb{Z}_2 & n = 2, 5, 7 \\
0 & \text{otherwise}
\end{cases} \]
A Cycle-Selecting Homomorphism $g$

- Denote the module generators of $H = H(A)$ by

$$
\begin{align*}
1 &= [1] \in H^0 \\
u &= [a_2] \in H^2 \\
v &= [a_2a_3 + a_3a_2] \in H^5 \\
w &= [a_2(a_2a_3 + a_3a_2)] \in H^7
\end{align*}
$$
A Cycle-Selecting Homomorphism $g$

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w = [a_2(a_2a_3 + a_3a_2)] \in H^7
\]

- Define a cycle-selecting homomorphism $g : H \to A$ by

\[
g(1) = 1 \\
g(u) = a_2 \\
g(v) = a_2a_3 + a_3a_2 \\
g(w) = a_2(a_2a_3 + a_3a_2)
\]
Suppose $g$ has a right-homotopy inverse $f$
g has no Right-Homotopy Inverse

- Suppose $g$ has a right-homotopy inverse $f$

- Then $gf(b_2) = \lambda a_2$ implies

$$b_2 - \lambda a_2 = (1 - gf)(b_2)$$
$$= (sd + ds)(b_2)$$
$$= sd(b_2) = s(2a_3) = 2s(a_3) = 0,$$

which is a contradiction
Suppose $g$ has a right-homotopy inverse $f$

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$$= (sd + ds) (b_2)$$
$$= sd (b_2) = s (2a_3) = 2s (a_3) = 0,$$

which is a contradiction

Transfer of structure in Example 1 cannot be computed using standard techniques
Induced DGA Structure

$\tilde{g} : \text{Hom}(H^\otimes 2, H) \to \text{Hom}(H^\otimes 2, A)$ is a quasi-isomorphism

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Induced DGA Structure

- $\tilde{g} : \text{Hom}(H \otimes 2, H) \rightarrow \text{Hom}(H \otimes 2, A)$ is a quasi-isomorphism

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- Evaluate $\tilde{g}$ on basis $\left\{ u \mid v \mapsto w, v \mid u \mapsto w \right\} \subset \text{Hom}^0(H \otimes 2, H)$ and evaluate $\Theta_2 = \mu(g \otimes g)$ on basis $\{u \mid v, v \mid u\}$ in degree 7

\[
\begin{align*}
    u \mid v \overset{w \partial_u \mid v}{\mapsto} w \overset{\tilde{g}}{\mapsto} a_2 (a_2 a_3 + a_3 a_2) &= \Theta_2 (u \mid v) \\
    v \mid u \overset{w \partial_v \mid u}{\mapsto} w \overset{\tilde{g}}{\mapsto} (a_2 a_3 + a_3 a_2) a_2 &= \Theta_2 (v \mid u)
\end{align*}
\]
Induced DGA Structure

- $\tilde{g} : \text{Hom} (H \otimes^2, H) \rightarrow \text{Hom} (H \otimes^2, A)$ is a quasi-isomorphism

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- Evaluate $\tilde{g}$ on basis $\left\{ u \cdot v \stackrel{w \partial_{u|v}}{\mapsto} w, \ v \cdot u \stackrel{w \partial_{v|u}}{\mapsto} w \right\} \subset \text{Hom}^0 (H \otimes^2, H)$ and evaluate $\Theta_2 = \mu (g \otimes g)$ on basis $\{u \cdot v, v \cdot u\}$ in degree 7

\[
\begin{align*}
 u \cdot v & \mapsto w \mapsto a_2 (a_2 a_3 + a_3 a_2) = \Theta_2 (u \cdot v) \\
 v \cdot u & \mapsto w \mapsto (a_2 a_3 + a_3 a_2) a_2 = \Theta_2 (v \cdot u)
\end{align*}
\]

- Thinking of $w \partial_{u|v} + w \partial_{v|u}$ as a class, $\tilde{g}_*(w \partial_{u|v} + w \partial_{v|u}) = [\Theta_2]$
Induced DGA Structure

\[ \tilde{g} : \text{Hom} \left( H \otimes^2, H \right) \rightarrow \text{Hom} \left( H \otimes^2, A \right) \]

is a quasi-isomorphism

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Evaluate \( \tilde{g} \) on basis \( \left\{ u|v \mapsto w, v|u \mapsto w \right\} \subseteq \text{Hom}^0 \left( H \otimes^2, H \right) \)

and evaluate \( \Theta_2 = \mu \left( g \otimes g \right) \) on basis \( \left\{ u|v, v|u \right\} \) in degree 7

\[ u|v \overset{\tilde{g}}{\mapsto} w \overset{w \partial_{u|v}}{\mapsto} a_2 \left( a_2 a_3 + a_3 a_2 \right) = \Theta_2 \left( u|v \right) \]

\[ v|u \overset{\tilde{g}}{\mapsto} w \overset{w \partial_{v|u}}{\mapsto} \left( a_2 a_3 + a_3 a_2 \right) a_2 = \Theta_2 \left( v|u \right) \]

Thinking of \( w \partial_{u|v} + w \partial_{v|u} \) as a class, \( \tilde{g}_* \left( w \partial_{u|v} + w \partial_{v|u} \right) = \left[ \Theta_2 \right] \)

Define \( \mu_H = \left( \tilde{g}_* \right)^{-1} \left[ \Theta_2 \right] = w \partial_{u|v} + w \partial_{v|u} \); then \( uv = vu = w \)
Extending $g$ to an $A(2)$-map

The non-trivial values of $\mu (g \otimes g) - g \mu_H$ are

$$a_2^2 \partial_{u|u}, \ (a_2^3 a_3 + a_3 a_2^3) \partial_{u|w}, \text{ and } (a_2^3 a_3 + a_3 a_2^3) \partial_{w|u}$$
Extending $g$ to an $A(2)$-map

- The non-trivial values of $\mu (g \otimes g) - g \mu_H$ are
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- These cocycles cobound

\[
\nabla (a_3 \partial_{u|u}) = da_3 \partial_{u|u} = a_2^2 \partial_{u|u} \\
\nabla (a_3 a_2 a_3 \partial_{u|w}) = d (a_3 a_2 a_3) \partial_{u|w} = (a_2^3 a_3 + a_3 a_2^3) \partial_{u|w} \\
\nabla (a_3 a_2 a_3 \partial_{w|u}) = d (a_3 a_2 a_3) \partial_{w|u} = (a_2^3 a_3 + a_3 a_2^3) \partial_{w|u}
\]
The non-trivial values of $\mu (g \otimes g) - g \mu_H$ are

\[
a_2^2 \partial_{u|_u}, \quad (a_2^3 a_3 + a_3 a_2^3) \partial_{u|w}, \quad \text{and} \quad (a_2^3 a_3 + a_3 a_2^3) \partial_{w|u}
\]

These cocycles cobound

\[
\nabla \left( a_3 \partial_{u|u} \right) = d a_3 \partial_{u|u} = a_2^2 \partial_{u|u} \\
\nabla \left( a_2 a_3 \partial_{u|w} \right) = d \left( a_2 a_3 \right) \partial_{u|w} = (a_2^3 a_3 + a_3 a_2^3) \partial_{u|w} \\
\nabla \left( a_2 a_3 \partial_{w|u} \right) = d \left( a_2 a_3 \right) \partial_{w|u} = (a_2^3 a_3 + a_3 a_2^3) \partial_{w|u}
\]

Thus we define

\[
g_2 = a_3 \partial_{u|u} + a_2 a_3 \left( \partial_{u|w} + \partial_{w|u} \right)
\]
Extending \( g \) to an \( A(2) \)-map

- The non-trivial values of \( \mu (g \otimes g) - g \mu_H \) are

\[
a_2^2 \partial_{u|u}, \quad (a_2^3 a_3 + a_3 a_2^3) \partial_{u|w}, \text{ and } (a_2^3 a_3 + a_3 a_2^3) \partial_{w|u}
\]

- These cocycles cobound

\[
\nabla (a_3 \partial_{u|u}) = da_3 \partial_{u|u} = a_2^2 \partial_{u|u} \\
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\nabla (a_3 a_2 a_3 \partial_{w|u}) = d (a_3 a_2 a_3) \partial_{w|u} = (a_2^3 a_3 + a_3 a_2^3) \partial_{w|u}
\]

- Thus we define

\[
g_2 = a_3 \partial_{u|u} + a_3 a_2 a_3 (\partial_{u|w} + \partial_{w|u})
\]

- Then \( \nabla g_2 = \mu (g \otimes g) - g \mu_H \) and \( g + g_2 \) is an \( A_2 \)-map
Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\Theta_3 = \mu (\mu_\mu \otimes g - g \otimes \mu_\mu) + \mu_\mu (\mu_H \otimes 1 - 1 \otimes \mu_H)$
Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\Theta_3 = \mu (g_2 \otimes g - g \otimes g_2) + g_2 (\mu_H \otimes 1 - 1 \otimes \mu_H)$

- $\bar{g} : \text{Hom} (H^\otimes 3, H) \rightarrow \text{Hom} (H^\otimes 3, A)$ is a quasi-isomorphism

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Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\bigotimes_3 = \mu \left( g_2 \otimes g - g \otimes g_2 \right) + g_2 \left( \mu_H \otimes 1 - 1 \otimes \mu_H \right)$

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- Evaluate $\tilde{g}$ on the basis $\left\{ u|u|u \right\}$ for $\text{Hom}^{-1} \left( H \otimes^3, H \right)$ and evaluate $\bigotimes_3$ on the basis $\left\{ u|u|u \right\}$ in degree 6

$$u|u|u \xrightarrow{v \partial_{u|u|u}} v \xrightarrow{\tilde{g}} a_2 a_3 + a_3 a_2 = \bigotimes_3 (u|u|u)$$
Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\Theta_3 = \mu (g_2 \otimes g - g \otimes g_2) + g_2 (\mu_H \otimes 1 - 1 \otimes \mu_H)$

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- Evaluate $\tilde{g}$ on the basis $\left\{ u|u|u \rightarrow^{v\partial_{u|u|u}} v \right\}$ for $\text{Hom}^{-1}(H^\otimes 3, H)$ and evaluate $\Theta_3$ on the basis $\{u|u|u\}$ in degree 6

$$u|u|u \overset{v\partial_{u|u|u}}{\rightarrow} v \overset{\tilde{g}}{\mapsto} a_2a_3 + a_3a_2 = \Theta_3(u|u|u)$$

- Thinking of $v\partial_{u|u|u}$ as a class: $\tilde{g}_*(v\partial_{u|u|u}) = [\Theta_3]$
Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\Theta_3 = \mu (g_2 \otimes g - g \otimes g_2) + g_2 (\mu_H \otimes 1 - 1 \otimes \mu_H)$

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- Evaluate $\bar{g}$ on the basis $\left\{ u|u|u \right\}$ for $Hom^{-1}(\mathbb{H}^3, H)$ and evaluate $\Theta_3$ on the basis $\{ u|u|u \}$ in degree 6

  $$u|u|u \quad \overset{v \partial_{u|u|u}}{\mapsto} \quad v \quad \overset{\bar{g} \quad a_2 a_3 + a_3 a_2}{\mapsto} \quad \Theta_3 (u|u|u)$$

- Thinking of $v \partial_{u|u|u}$ as a class: $\bar{g}_* \left( v \partial_{u|u|u} \right) = \left[ \Theta_3 \right]$

- Define $\mu^3_H = (\bar{g}_*)^{-1} \left[ \Theta_3 \right] = v \partial_{u|u|u}$; then $\mu^3_H (u|u|u) = v$
Induced Higher Order Structure

\[ \Theta_3 - g\mu_3^H \equiv 0 \]
\[ \Theta_3 - g \mu_3^H \equiv 0 \]

- Define \( g_n = 0 \) and \( \mu^n_H = 0 \) for all \( n \geq 4 \)
Induced Higher Order Structure

- $\Theta_3 - g \mu^3_H \equiv 0$

- Define $g_n = 0$ and $\mu^n_H = 0$ for all $n \geq 4$

- Then $(H, \mu_H, \mu^3_H)$ is an $A_\infty$-algebra and $G = g + g_2$ is $A_\infty$-map
\[ \Theta_3 - g \mu_H^3 \equiv 0 \]

Define \( g_n = 0 \) and \( \mu^n_H = 0 \) for all \( n \geq 4 \)

Then \((H, \mu_H, \mu^3_H)\) is an \( A_\infty \)-algebra and \( G = g + g_2 \) is \( A_\infty \)-map

Theorem 1 extends immediately to \( A_\infty \)-bialgebras
Transfer is controlled by generalized multiplihedra $\{JJ_{m,n}\}_{m,n \geq 1}$ of which $JJ_{n,1} = JJ_{1,n} = J_n$

$$(g \otimes g) \omega_B^{2,2}$$

The Generalized Multiplihedron $JJ_{2,2}$
Transfer Problem 2

- The problem is exactly the same as before
Transfer Problem 2

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- Identify $g_{n,m}$ with the $(m + n - 1)$-dimensional polyhedron $J_{n,m}$
Transfer Problem 2

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- Identify $g_{n,m}$ with the $(m + n - 1)$-dimensional polyhedron $J_{n,m}$
- Let $Q^{m+n-2} \subset JJ_{n,m}$ denote the cell identified with $g \otimes n \omega_{B}^{n,m}$
Transfer Problem 2

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- $\bigotimes_m^n \in Hom^{m+n-2}(B \otimes m, A \otimes n)$ is identified with $\partial JJ_{n,m} - \text{int } Q^{m+n-2}$
Transfer Problem 2

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- Let $Q^{m+n-2} \subset JJ_{n,m}$ denote the cell identified with $g \otimes^n \omega_B^{n,m}$
- $\ominus^n_m \in \text{Hom}^{m+n-2} (B \otimes^m, A \otimes^n)$ is identified with $\partial JJ_{n,m} - \text{int} Q^{m+n-2}$
- $\nabla \ominus^n_m \in \text{Hom}^{m+n-3} (B \otimes^m, A \otimes^n)$ is identified with $\partial Q^{m+n-2}$
Transfer Problem 2

- The problem is exactly the same as before

- Identify \( g_{n,m} \) with the \((m + n - 1)\)-dimensional polyhedron \( J_{n,m} \)

- Let \( Q^{m+n-2} \subset J_{n,m} \) denote the cell identified with \( g \otimes \omega_B^n \)

- \( \ominus^n_m \in \text{Hom}^{m+n-2}(B \otimes^m, A \otimes^n) \) is identified with \( \partial J_{n,m} - \text{int} Q^{m+n-2} \)

- \( \nabla \ominus^n_m \in \text{Hom}^{m+n-3}(B \otimes^m, A \otimes^n) \) is identified with \( \partial Q^{m+n-2} \)

**Transfer Problem 2:** Let \( A \) be an \( A_\infty \)-bialgebra, let \( B \) be a DGM, and let \( g : B \to A \) be a chain map. Given \( \{ \omega_B^{j,i}, g_i^j \}_{1 \leq i+j<k} \) construct \( g^n_m \) and \( \omega_B^{n,m} \) for each \((m, n)\) with \( m + n = k \) so that

\[
\nabla g^n_m = \ominus^n_m - g \otimes \omega_B^n \]

A chain map $g$ induces a cochain map

$$\tilde{g} : \left( \text{Hom} \left( B^\otimes m, B^\otimes n \right); \nabla_B \right) \to \text{Hom} \left( B^\otimes m, A^\otimes n; \nabla \right)$$

given by $\tilde{g}(u) = g^\otimes n u$
Second Transfer Theorem

- A chain map $g$ induces a cochain map

$$\tilde{g} : \left( \text{Hom} \left( B^\otimes m, B^\otimes n \right); \nabla_B \right) \to \text{Hom} \left( B^\otimes m, A^\otimes n; \nabla \right)$$

given by $\tilde{g} \left( u \right) = g^\otimes n u$

- Transfer Problem 2 has a solution whenever $\tilde{g}$ is a quasi-isomorphism
Second Transfer Theorem

- A chain map $g$ induces a cochain map

$$\tilde{g} : \left( \text{Hom} \left( B^\otimes m, B^\otimes n \right); \nabla_B \right) \rightarrow \text{Hom} \left( B^\otimes m, A^\otimes n; \nabla \right)$$

given by $\tilde{g}(u) = g^\otimes n u$

- Transfer Problem 2 has a solution whenever $\tilde{g}$ is a quasi-isomorphism

**Theorem 2.** If $\tilde{g}$ is a quasi-isomorphism, then

(i) $g$ transfers the $A_\infty$-bialgebra structure from $A$ to $B$; the induced structure on $B$ is unique up to automorphism

(ii) $g$ extends to a map $G : B \Rightarrow A$ of $A_\infty$-bialgebras
Proposition 2. Let $A$ be an $A_\infty$-bialgebra $A$ over a field $k$, and choose a cycle-selecting homomorphism $g : H \to A$. Then $\tilde{g}$ is a quasi-isomorphism.
Proposition 2. Let $A$ be an $A_{\infty}$-bialgebra $A$ over a field $k$, and choose a cycle-selecting homomorphism $g : H \to A$. Then $\tilde{g}$ is a quasi-isomorphism.

Corollary 2. If $A$ is an $A_{\infty}$-bialgebra over a field $k$, there is an induced $A_{\infty}$-bialgebra structure on $H(A; k)$.
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Compatibility of $\mu$ with $\Delta^3$ is expressed by the relation

$$\Delta^3 \mu = \mu^\otimes^3 \sigma_{3,2} \left[ (\Delta \otimes 1) \Delta \otimes \Delta^3 + \Delta^3 \otimes (1 \otimes \Delta) \Delta \right],$$

where $\sigma_{p,q} : (H^\otimes p)^\otimes q \rightarrow (H^\otimes q)^\otimes p$ permutes tensor factors
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Then $\zeta(e^{n-2}) = \Delta^n$ and

$$\Delta^n \mu = \mu^{\otimes n} \sigma_{n,2} \left[ (\zeta \otimes \zeta) \Delta_K (e^{n-2}) \right]$$
Thank you!