Transferring A_∞-Structures from Chains to Homology

Joint work with Samson Saneblidze

Ron Umble
Univ de Sevilla and Millersville Univ of PA

4D Digital Imaging Seminar

24 September 2009

Goal of the Talk

To understand the following statement:

- **Theorem.** *If* A *is an* A_∞*-structure over a field* $\mathbf{k}$*, there is an induced* A_∞*-structure on* $H(A;\mathbf{k})$.

Goal of the Talk

To understand the following statement:

- **Theorem.** *If A is an A_∞-structure over a field $\mathbf{k}$, there is an induced A_∞-structure on $H(A; \mathbf{k})$.*

- Many have studied the transfer of A_∞-algebra structure, including Kadeishvili, Huebschmann, Kontsevich, Soibelman, Merkulov, Markl, Real, M-J Jiménez, Berciano, to name a few...

Goal of the Talk

To understand the following statement:

- **Theorem.** *If A is an A_∞-structure over a field* $\mathbf{k}$, *there is an induced A_∞-structure on* $H(A;\mathbf{k})$.

- Many have studied the transfer of A_∞-algebra structure, including Kadeishvili, Huebschmann, Kontsevich, Soibelman, Merkulov, Markl, Real, M-J Jiménez, Berciano, to name a few...

- Saneblidze observed that the "Transfer Problem" is simpler at the level of hom groups

Goal of the Talk

To understand the following statement:

- **Theorem.** *If A is an A_∞-structure over a field* $\mathbf{k}$, *there is an induced A_∞-structure on* $H(A;\mathbf{k})$.
- Many have studied the transfer of A_∞-algebra structure, including Kadeishvili, Huebschmann, Kontsevich, Soibelman, Merkulov, Markl, Real, M-J Jiménez, Berciano, to name a few...
- Saneblidze observed that the "Transfer Problem" is simpler at the level of hom groups
- Our method relaxes the conditions under which the transfer of A_∞-algebra structure occurs, and transfers A_∞-bialgebra structure as well

Background: A-infinity Algebras

- Cellular chains on associahedra $K = \{K_n\}_{n \geq 2}$ realize the operad $\mathcal{A}_\infty$

Background: A-infinity Algebras

- Cellular chains on associahedra $K = \{K_n\}_{n \geq 2}$ realize the operad $\mathcal{A}_\infty$
- $\dim (K_n) = n - 2$

Background: A-infinity Algebras

- Cellular chains on associahedra $K = \{K_n\}_{n \geq 2}$ realize the operad $\mathcal{A}_\infty$
- $\dim (K_n) = n - 2$
- An A_∞-*algebra* is a DGM (A, d) together with a family of operations

$$\{\mu^n \in Hom^{n-2}(A^{\otimes n}, A)\}_{n \geq 2}$$

and a map $\varphi : \mathcal{A}_\infty \to \{Hom(A^{\otimes n}, A)\}_{n \geq 1}$ of non-Σ operads

Background: A-infinty Bialgebras

- Cellular chains on matrahedra $KK = \{KK_{n,m}\}$ realize the *matrad* $\mathcal{H}_\infty$

Background: A-infinty Bialgebras

- Cellular chains on matrahedra $KK = \{KK_{n,m}\}$ realize the *matrad* $\mathcal{H}_\infty$
- $\dim(KK_{n,m}) = m + n - 3$

Background: A-infinty Bialgebras

- Cellular chains on matrahedra $KK = \{KK_{n,m}\}$ realize the *matrad* $\mathcal{H}_\infty$
- $\dim(KK_{n,m}) = m + n - 3$
- An A_∞-*bialgebra* is DGM (H, d) together with a family of operations

$$\left\{\omega_m^n \in Hom^{m+n-3}\left(H^{\otimes m}, H^{\otimes n}\right)\right\}_{m,n\geq 1;\ m+n\geq 3}$$

and a map $\varphi : \mathcal{H}_\infty \to End(TH)$ of matrads

Introduction and Overview

- Given DGMs (A, d_A) and (B, d_B), define ∇ on $Hom(B, A)$ by

$$\nabla(u) = d_A u - (-1)^{|u|} u d_B$$

Introduction and Overview

- Given DGMs (A, d_A) and (B, d_B), define ∇ on $Hom(B, A)$ by

$$\nabla(u) = d_A u - (-1)^{|u|} u d_B$$

- A chain map $g : B \to A$ induces a cochain map

$$\bar{g} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B\right) \to \left(Hom\left(B^{\otimes n}, A\right); \nabla\right)$$

via $g(u) = gu$

Introduction and Overview

- Given DGMs (A, d_A) and (B, d_B), define ∇ on $Hom(B, A)$ by

$$\nabla(u) = d_A u - (-1)^{|u|} u d_B$$

- A chain map $g : B \to A$ induces a cochain map

$$\bar{g} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B\right) \to \left(Hom\left(B^{\otimes n}, A\right); \nabla\right)$$

  via $g(u) = gu$

- **Theorem 1.** *Let A be an A_∞-algebra, B be a* DGM, *and $g : B \to A$ be a chain map. If $\bar{g}$ is a quasi-isomorphism, then g induces an A_∞-algebra structure on B.*

Introduction and Overview

- Given DGMs (A, d_A) and (B, d_B), define ∇ on $Hom(B, A)$ by

$$\nabla(u) = d_A u - (-1)^{|u|} u d_B$$

- A chain map $g : B \longrightarrow A$ induces a cochain map

$$\bar{g} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B\right) \longrightarrow \left(Hom\left(B^{\otimes n}, A\right); \nabla\right)$$

  via $g(u) = gu$

- **Theorem 1.** *Let A be an A_∞-algebra, B be a* DGM, *and $g : B \longrightarrow A$ be a chain map. If $\bar{g}$ is a quasi-isomorphism, then g induces an A_∞-algebra structure on B.*

- If $\bar{g}$ is a quasi-isomorphism, so is g, but not conversely

Introduction and Overview

- Given DGMs (A, d_A) and (B, d_B), define ∇ on $Hom(B, A)$ by

$$\nabla(u) = d_A u - (-1)^{|u|} u d_B$$

- A chain map $g : B \to A$ induces a cochain map

$$\bar{g} : \left(Hom\left(B^{\otimes n}, B\right); \nabla_B\right) \to \left(Hom\left(B^{\otimes n}, A\right); \nabla\right)$$

  via $g(u) = gu$

- **Theorem 1.** *Let A be an A_∞-algebra, B be a* DGM, *and $g : B \to A$ be a chain map. If $\bar{g}$ is a quasi-isomorphism, then g induces an A_∞-algebra structure on B.*

- If $\bar{g}$ is a quasi-isomorphism, so is g, but not conversely

- However, the converse holds whenever B is free

Introduction and Overview

- Kadeishvili and others assume B is free

Introduction and Overview

- Kadeishvili and others assume B is free
- Markl assumes g has a right-homotopy inverse f

$$1 - gf = d_A\phi + \phi d_A$$

Introduction and Overview

- Kadeishvili and others assume B is free
- Markl assumes g has a right-homotopy inverse f

$$1 - gf = d_A \phi + \phi d_A$$

- All assume there is a homotopy operator $\phi : A \to A$

Introduction and Overview

- Kadeishvili and others assume B is free
- Markl assumes g has a right-homotopy inverse f

$$1 - gf = d_A \phi + \phi d_A$$

- All assume there is a homotopy operator $\phi : A \to A$
- Our algorithm requires neither freeness nor a homotopy operator

Introduction and Overview

- Kadeishvili and others assume B is free
- Markl assumes g has a right-homotopy inverse f

$$1 - gf = d_A\phi + \phi d_A$$

- All assume there is a homotopy operator $\phi : A \to A$
- Our algorithm requires neither freeness nor a homotopy operator
- $B = H(A)$ is an important special case of interest in this talk

Key Points in the Talk

- **Corollary 1.** *Let A be an A_∞-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is an induced A_∞-algebra structure on H.*

Key Points in the Talk

- **Corollary 1.** *Let A be an A_∞-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is an induced A_∞-algebra structure on H.*

- **Example 1.** We exhibit a DGA A with an induced non-trivial A_∞-algebra structure on $H(A)$ that cannot be computed using standard techniques

Key Points in the Talk

- **Corollary 1.** *Let A be an A_∞-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is an induced A_∞-algebra structure on H.*

- **Example 1.** We exhibit a DGA A with an induced non-trivial A_∞-algebra structure on $H(A)$ that cannot be computed using standard techniques

- **Corollary 2.** *Given an A_∞-bialgebra A over a field* $\mathbf{k}$*, there is an induced A_∞-bialgebra structure on H*

Key Points in the Talk

- **Corollary 1.** *Let A be an A_∞-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is an induced A_∞-algebra structure on H.*

- **Example 1.** We exhibit a DGA A with an induced non-trivial A_∞-algebra structure on $H(A)$ that cannot be computed using standard techniques

- **Corollary 2.** *Given an A_∞-bialgebra A over a field* **k**, *there is an induced A_∞-bialgebra structure on H*

- **Example 2.** *Let X be a space. There is an induced A_∞-bialgebra structure on $H_*(\Omega X; \mathbf{k})$*

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $g : B \to A$, a morphism

$$G = g + g_2 + g_3 + \cdots : B \Rightarrow A$$

is defined in terms of parameter spaces $\{J_n\}_{n\geq 1}$, called *multiplihedra*

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $g : B \to A$, a morphism

$$G = g + g_2 + g_3 + \cdots : B \Rightarrow A$$

  is defined in terms of parameter spaces $\{J_n\}_{n \geq 1}$, called *multiplihedra*

- $J_1 = *$ is identified with the cochain $g \in Hom^0(B, A)$

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $g : B \to A$, a morphism

$$G = g + g_2 + g_3 + \cdots : B \Rightarrow A$$

  is defined in terms of parameter spaces $\{J_n\}_{n \geq 1}$, called *multiplihedra*

- $J_1 = *$ is identified with the cochain $g \in Hom^0(B, A)$
- $J_2 = I$ is identified with the cochain $g_2 \in Hom^1(B \otimes B, A)$

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $g : B \to A$, a morphism

$$G = g + g_2 + g_3 + \cdots : B \Rightarrow A$$

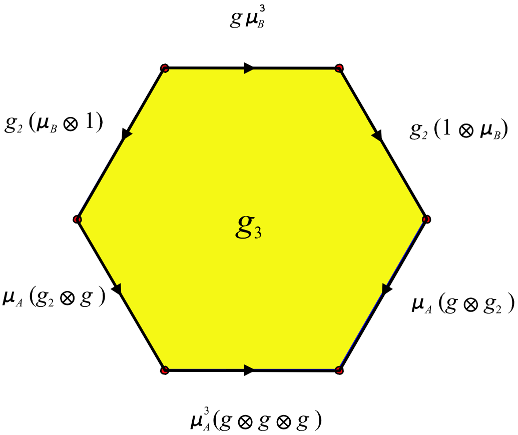
is defined in terms of parameter spaces $\{J_n\}_{n \geq 1}$, called *multiplihedra*

- $J_1 = *$ is identified with the cochain $g \in Hom^0(B, A)$
- $J_2 = I$ is identified with the cochain $g_2 \in Hom^1(B \otimes B, A)$
- The endpoints of J_2 are identified with components of the coboundary

$$\nabla g_2 = \mu_A (g \otimes g) - g\mu_B$$

A–infinity Maps and Multiplihedra

J_3 is an hexagonal plane region identified with $g_3 \in Hom^2\left(B^{\otimes 3}, A\right)$



$$\nabla g_3 = \mu_A^3 g^{\otimes 3} + \mu_A\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) - g\mu_B^3$$

A–infinity Maps and Multiplihedra

- J_n is $(n-1)$-dimensional and identified with $g_n \in Hom^{n-1}\left(B^{\otimes n}, A\right)$

A–infinity Maps and Multiplihedra

- J_n is $(n-1)$-dimensional and identified with $g_n \in Hom^{n-1}\left(B^{\otimes n}, A\right)$
- The operations μ_B^k that appear in ∇g_n have order $k \leq n$

A–infinity Maps and Multiplihedra

- J_n is $(n-1)$-dimensional and identified with $g_n \in Hom^{n-1}\left(B^{\otimes n}, A\right)$
- The operations μ_B^k that appear in ∇g_n have order $k \leq n$
- μ_B^n appears in exactly one component of ∇g_n, namely $g\mu_B^n$

A–infinity Maps and Multiplihedra

- J_n is $(n-1)$-dimensional and identified with $g_n \in Hom^{n-1}\left(B^{\otimes n}, A\right)$
- The operations μ_B^k that appear in ∇g_n have order $k \leq n$
- μ_B^n appears in exactly one component of ∇g_n, namely $g\mu_B^n$
- Let Q^{n-2} denote the cell of J_n identified with the cochain $g\mu_B^n$

A–infinity Maps and Multiplihedra

- J_n is $(n-1)$-dimensional and identified with $g_n \in Hom^{n-1}\left(B^{\otimes n}, A\right)$
- The operations μ_B^k that appear in ∇g_n have order $k \leq n$
- μ_B^n appears in exactly one component of ∇g_n, namely $g\mu_B^n$
- Let Q^{n-2} denote the cell of J_n identified with the cochain $g\mu_B^n$
- Let $\ominus_n$ denote the cochain identified with $\partial J_n - \operatorname{int} Q^{n-2}$

A–infinity Maps and Multiplihedra

- J_n is $(n-1)$-dimensional and identified with $g_n \in Hom^{n-1}\left(B^{\otimes n}, A\right)$
- The operations μ_B^k that appear in ∇g_n have order $k \leq n$
- μ_B^n appears in exactly one component of ∇g_n, namely $g\mu_B^n$
- Let Q^{n-2} denote the cell of J_n identified with the cochain $g\mu_B^n$
- Let $\ominus_n$ denote the cochain identified with $\partial J_n - \operatorname{int} Q^{n-2}$
- Then ∂Q^{n-2} is identified with the coboundary $\nabla \ominus_n$

First Transfer Theorem

- **Transfer Problem 1:** *Let* A *be an* A_∞*-algebra, let* B *be a* DGM,*and let* $g : B \to A$ *be a chain map. Given* $\{\mu_B^i,\ g_i\}_{2\leq i<n}$ *construct* μ_B^n *and* g_n *so that*

$$\nabla g_n = \ominus_n - g\mu_B^n$$

First Transfer Theorem

- **Transfer Problem 1:** *Let* A *be an* A_∞*-algebra, let* B *be a* DGM,*and let* $g : B \to A$ *be a chain map. Given* $\{\mu_B^i, g_i\}_{2 \leq i < n}$ *construct* μ_B^n *and* g_n *so that*
$$\nabla g_n = \ominus_n - g\mu_B^n$$
- Transfer Problem 1 has a solution whenever $\bar{g}$ is a quasi-isomorphism

First Transfer Theorem

- **Transfer Problem 1:** *Let* A *be an* A_∞*-algebra, let* B *be a* DGM,*and let* $g : B \to A$ *be a chain map. Given* $\{\mu_B^i, g_i\}_{2\leq i<n}$ *construct* μ_B^n *and* g_n *so that*
$$\nabla g_n = \ominus_n - g\mu_B^n$$
- Transfer Problem 1 has a solution whenever $\bar{g}$ is a quasi-isomorphism
- **Theorem 1**. *If* $\bar{g}$ *is a quasi-isomorphism, then*

  (i) g *transfers the* A_∞*-algebra structure from* A *to* B*; the induced structure on* B *is unique up to automorphism.*

  (ii) g *extends to a map* $G : B \Rightarrow A$ *of* A_∞*-algebras.*

Key Proposition

- **Proposition 1.** *Let A be an A_∞-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is a cycle-selecting homomorphism $g : H \longrightarrow A$ such that $\bar{g}$ is a quasi-isomorphism.*

Key Proposition

- **Proposition 1.** *Let A be an A_∞-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is a cycle-selecting homomorphism $g : H \to A$ such that $\bar{g}$ is a quasi-isomorphism.*

- **Corollary 1.** *Let A be an A_∞-algebra and let $H = H(A)$. If $A = H \oplus X$ and $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is an induced A_∞-algebra structure on H.*

Computing the Transfer: The Fundamental Cocycle

Let $(A, d, \mu^n)_{n \geq 2}$ be an A_∞-algebra, $g : H \to A$ a cycle-selecting hom, and assume that an A_∞-structure $(H, \mu_H^n)_{n \geq 2}$ has been constructed

- Thinking of $g \in Hom(H, A)$, note that $\nabla g = dg = 0$

Computing the Transfer: The Fundamental Cocycle

Let $(A, d, \mu^n)_{n\geq 2}$ be an A_∞-algebra, $g : H \longrightarrow A$ a cycle-selecting hom, and assume that an A_∞-structure $(H, \mu_H^n)_{n\geq 2}$ has been constructed

- Thinking of $g \in Hom\,(H, A)$, note that $\nabla g = dg = 0$
- $\ominus_2 = \mu\,(g \otimes g) \in Hom^0\,(H \otimes H, A)$

Computing the Transfer: The Fundamental Cocycle

Let $(A, d, \mu^n)_{n\geq 2}$ be an A_∞-algebra, $g : H \longrightarrow A$ a cycle-selecting hom, and assume that an A_∞-structure $(H, \mu_H^n)_{n\geq 2}$ has been constructed

- Thinking of $g \in Hom(H, A)$, note that $\nabla g = dg = 0$
- $\ominus_2 = \mu(g \otimes g) \in Hom^0(H \otimes H, A)$
- $\nabla \ominus_2 = d\mu(g \otimes g) = \mu(dg \otimes g + g \otimes dg) = 0$

Computing the Transfer: The Fundamental Cocycle

- For $n \geq 3$, recall that $g\mu_H^n$ is identified with Q^{n-2}

Computing the Transfer: The Fundamental Cocycle

- For $n \geq 3$, recall that $g\mu_H^n$ is identified with Q^{n-2}
- $\ominus_n \in Hom^{n-2}\left(H^{\otimes n}, A\right)$ is identified with $\partial J_n - \operatorname{int} Q^{n-2}$

Computing the Transfer: The Fundamental Cocycle

- For $n \geq 3$, recall that $g\mu_H^n$ is identified with Q^{n-2}
- $\ominus_n \in Hom^{n-2}\left(H^{\otimes n}, A\right)$ is identified with $\partial J_n - \operatorname{int} Q^{n-2}$
- Since $\partial\left(\partial J_n - \operatorname{int} Q^{n-2}\right) = \partial Q^{n-2}$ we have

Computing the Transfer: The Fundamental Cocycle

- For $n \geq 3$, recall that $g\mu_H^n$ is identified with Q^{n-2}
- $\ominus_n \in Hom^{n-2}\left(H^{\otimes n}, A\right)$ is identified with $\partial J_n - \operatorname{int} Q^{n-2}$
- Since $\partial\left(\partial J_n - \operatorname{int} Q^{n-2}\right) = \partial Q^{n-2}$ we have
- $\nabla \ominus_n = \nabla g\mu_H^n = dg\mu_H^n = 0$

Computing the Transfer: The Fundamental Cocycle

- For $n \geq 3$, recall that $g\mu_H^n$ is identified with Q^{n-2}
- $\ominus_n \in Hom^{n-2}\left(H^{\otimes n}, A\right)$ is identified with $\partial J_n - \text{int}\, Q^{n-2}$
- Since $\partial\left(\partial J_n - \text{int}\, Q^{n-2}\right) = \partial Q^{n-2}$ we have
- $\nabla \ominus_n = \nabla g\mu_H^n = dg\mu_H^n = 0$
- Therefore $\ominus_n$ is a cocycle for all $n \geq 2$

Computing the Transfer: The Fundamental Cocycle

- For $n \geq 3$, recall that $g\mu_H^n$ is identified with Q^{n-2}
- $\ominus_n \in Hom^{n-2}\left(H^{\otimes n}, A\right)$ is identified with $\partial J_n - \text{int}\, Q^{n-2}$
- Since $\partial\left(\partial J_n - \text{int}\, Q^{n-2}\right) = \partial Q^{n-2}$ we have
- $\nabla \ominus_n = \nabla g\mu_H^n = dg\mu_H^n = 0$
- Therefore $\ominus_n$ is a cocycle for all $n \geq 2$
- We refer to $\ominus_n$ as the *fundamental n-cocycle*

Computing the Transfer: The Induction

Now suppose $\bar{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\left\{\mu_H^i, g_i\right\}_{2 \leq i < n}$ has been constructed

Computing the Transfer: The Induction

Now suppose $\bar{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu_H^i, g_i\}_{2 \leq i < n}$ has been constructed
- Since $\nabla \ominus_n = 0$, we may define $\mu_H^n = (\bar{g}_*)^{-1} [\ominus_n]$

Computing the Transfer: The Induction

Now suppose $\bar{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu_H^i, g_i\}_{2 \leq i < n}$ has been constructed
- Since $\nabla \ominus_n = 0$, we may define $\mu_H^n = (\bar{g}_*)^{-1} [\ominus_n]$
- Then $g\mu_H^n \in [\ominus_n]$ and $\ominus_n - g\mu_H^n$ is a coboundary

Computing the Transfer: The Induction

Now suppose $\bar{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu_H^i, g_i\}_{2 \leq i < n}$ has been constructed
- Since $\nabla \ominus_n = 0$, we may define $\mu_H^n = (\bar{g}_*)^{-1} [\ominus_n]$
- Then $g\mu_H^n \in [\ominus_n]$ and $\ominus_n - g\mu_H^n$ is a coboundary
- Solve the linear system $dx = \ominus_n - g\mu_H^n$

Computing the Transfer: The Induction

Now suppose $\bar{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu_H^i, g_i\}_{2 \leq i < n}$ has been constructed
- Since $\nabla \ominus_n = 0$, we may define $\mu_H^n = (\bar{g}_*)^{-1} [\ominus_n]$
- Then $g\mu_H^n \in [\ominus_n]$ and $\ominus_n - g\mu_H^n$ is a coboundary
- Solve the linear system $dx = \ominus_n - g\mu_H^n$
- Choose a particular solution g_n

Computing the Transfer: The Induction

Now suppose $\bar{g}$ is a quasi-isomorphism

- For $n \geq 2$, assume that $\{\mu_H^i, g_i\}_{2 \leq i < n}$ has been constructed
- Since $\nabla \ominus_n = 0$, we may define $\mu_H^n = (\bar{g}_*)^{-1} [\ominus_n]$
- Then $g\mu_H^n \in [\ominus_n]$ and $\ominus_n - g\mu_H^n$ is a coboundary
- Solve the linear system $dx = \ominus_n - g\mu_H^n$
- Choose a particular solution g_n
- Then $g + g_2 + \cdots + g_n$ is an A_n-map

Transfer of A-infinity Algebra Structure

- **Example 1.** Consider the DGM

$$
\begin{array}{ccccccccccccc}
M^0 & \to & 0 & \to & M^2 & \to & M^3 & \to & M^4 & \to & 0 & \to & \cdots \\
\\
\mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\
 & & & & (a_2, b_2) & \mapsto & (0, 2a_3) & & & & & & \\
 & & & & & & a_3 & \mapsto & a_4 & & & &
\end{array}
$$

- **Example 1.** Consider the DGM

$$
\begin{array}{ccccccccccccc}
M^0 & \to & 0 & \to & M^2 & \to & M^3 & \to & M^4 & \to & 0 & \to & \cdots \\
\mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\
 & & & & (a_2, b_2) & \mapsto & (0, 2a_3) & & & & & & \\
 & & & & & & a_3 & \mapsto & a_4 & & & &
\end{array}
$$

- $A = T^a M \,/\, \left(a_2^2 + a_4, a_3^2, a_4 a_3 + a_3 a_4, (a_2 a_3 + a_3 a_2)^2, a_i b_2, b_2 a_i, b_2^2\right)$

Transfer of A-infinity Algebra Structure

- **Example 1.** Consider the DGM

$$\begin{array}{ccccccccccccc} M^0 & \to & 0 & \to & M^2 & \to & M^3 & \to & M^4 & \to & 0 & \to & \cdots \\ \\ \mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\ & & & & (a_2, b_2) & \mapsto & (0, 2a_3) & & & & & & \\ & & & & & & a_3 & \mapsto & a_4 & & & & \end{array}$$

- $A = T^a M \,/\, \left(a_2^2 + a_4, a_3^2, a_4 a_3 + a_3 a_4, (a_2 a_3 + a_3 a_2)^2, a_i b_2, b_2 a_i, b_2^2\right)$

- A has no Hodge decomposition since $\mathbb{Z}_4$ contains a non-cycle a_3 and a boundary $2a_3$. But $\mathbb{Z}_4$ does not split as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Transfer of A-infinity Algebra Structure

- **Example 1.** Consider the DGM

$$
\begin{array}{ccccccccccccc}
M^0 & \to & 0 & \to & M^2 & \to & M^3 & \to & M^4 & \to & 0 & \to & \cdots \\
\mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\
 & & & & (a_2, b_2) & \mapsto & (0, 2a_3) & & & & & & \\
 & & & & & & a_3 & \mapsto & a_4 & & & &
\end{array}
$$

- $A = T^a M \,/\, \left(a_2^2 + a_4, a_3^2, a_4 a_3 + a_3 a_4, (a_2 a_3 + a_3 a_2)^2, a_i b_2, b_2 a_i, b_2^2\right)$

- A has no Hodge decomposition since $\mathbb{Z}_4$ contains a non-cycle a_3 and a boundary $2a_3$. But $\mathbb{Z}_4$ does not split as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

- $H^n(A) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2, 5, 7 \\ 0 & \text{otherwise} \end{cases}$

A Cycle-Selecting Homomorphism g

- Denote the module generators of $H = H(A)$ by

$$\begin{aligned} 1 &= [1] \in H^0 \\ u &= [a_2] \in H^2 \\ v &= [a_2 a_3 + a_3 a_2] \in H^5 \\ w &= [a_2(a_2 a_3 + a_3 a_2)] \in H^7 \end{aligned}$$

A Cycle-Selecting Homomorphism g

- Denote the module generators of $H = H(A)$ by

$$\begin{aligned} 1 &= [1] \in H^0 \\ u &= [a_2] \in H^2 \\ v &= [a_2 a_3 + a_3 a_2] \in H^5 \\ w &= [a_2(a_2 a_3 + a_3 a_2)] \in H^7 \end{aligned}$$

- Define a cycle-selecting homomoprhism $g : H \to A$ by

$$\begin{aligned} g(1) &= 1 \\ g(u) &= a_2 \\ g(v) &= a_2 a_3 + a_3 a_2 \\ g(w) &= a_2(a_2 a_3 + a_3 a_2) \end{aligned}$$

g has no Right-Homotopy Inverse

- Suppose g has a right-homotopy inverse f

g has no Right-Homotopy Inverse

- Suppose g has a right-homotopy inverse f
- Then $gf(b_2) = \lambda a_2$ implies

$$
\begin{aligned}
b_2 - \lambda a_2 &= (1 - gf)(b_2) \\
&= (sd + ds)(b_2) \\
&= sd(b_2) = s(2a_3) = 2s(a_3) = 0,
\end{aligned}
$$

which is a contradiction

g has no Right-Homotopy Inverse

- Suppose g has a right-homotopy inverse f
- Then $gf(b_2) = \lambda a_2$ implies

$$\begin{aligned} b_2 - \lambda a_2 &= (1 - gf)(b_2) \\ &= (sd + ds)(b_2) \\ &= sd(b_2) = s(2a_3) = 2s(a_3) = 0, \end{aligned}$$

  which is a contradiction

- Transfer of structure in Example 1 cannot be computed using standard techniques

Induced DGA Structure

- $\bar{g} : Hom\left(H^{\otimes 2}, H\right) \to Hom\left(H^{\otimes 2}, A\right)$ is a quasi-isomorphism

	2	4	5	7	9	10	12	14
H	u		v	w				
$H \otimes H$		$u\|u$		$u\|v,\ v\|u$	$u\|w,\ w\|u$	$v\|v$	$v\|w,\ w\|v$	$w\|w$

Induced DGA Structure

- $\bar{g} : Hom\left(H^{\otimes 2}, H\right) \to Hom\left(H^{\otimes 2}, A\right)$ is a quasi-isomorphism

	2	4	5	7	9	10	12	14
H	u		v	w				
$H \otimes H$		$u\|u$		$u\|v,\ v\|u$	$u\|w,\ w\|u$	$v\|v$	$v\|w,\ w\|v$	$w\|w$

- Evaluate $\bar{g}$ on basis $\left\{ u|v \overset{w\partial_{u|v}}{\mapsto} w,\ v|u \overset{w\partial_{v|u}}{\mapsto} w \right\} \subset Hom^0\left(H^{\otimes 2}, H\right)$ and evaluate $\ominus_2 = \mu\left(g \otimes g\right)$ on basis $\{u|v, v|u\}$ in degree 7

$$u|v \overset{w\partial_{u|v}}{\mapsto} w \overset{g}{\mapsto} a_2\left(a_2a_3 + a_3a_2\right) = \ominus_2\left(u|v\right)$$

$$v|u \overset{w\partial_{v|u}}{\mapsto} w \overset{g}{\mapsto} \left(a_2a_3 + a_3a_2\right)a_2 = \ominus_2\left(v|u\right)$$

Induced DGA Structure

- $\bar{g} : Hom\left(H^{\otimes 2}, H\right) \to Hom\left(H^{\otimes 2}, A\right)$ is a quasi-isomorphism

	2	4	5	7	9	10	12	14
H	u		v	w				
$H \otimes H$		$u\|u$		$u\|v,\ v\|u$	$u\|w,\ w\|u$	$v\|v$	$v\|w,\ w\|v$	$w\|w$

- Evaluate $\bar{g}$ on basis $\left\{ u|v \overset{w\partial_{u|v}}{\mapsto} w,\ v|u \overset{w\partial_{v|u}}{\mapsto} w \right\} \subset Hom^0\left(H^{\otimes 2}, H\right)$ and evaluate $\ominus_2 = \mu\left(g \otimes g\right)$ on basis $\{u|v, v|u\}$ in degree 7

$$u|v \overset{w\partial_{u|v}}{\mapsto} w \overset{g}{\mapsto} a_2\left(a_2 a_3 + a_3 a_2\right) = \ominus_2\left(u|v\right)$$
$$v|u \overset{w\partial_{v|u}}{\mapsto} w \overset{g}{\mapsto} \left(a_2 a_3 + a_3 a_2\right) a_2 = \ominus_2\left(v|u\right)$$

- Thinking of $w\partial_{u|v} + w\partial_{v|u}$ as a class, $\bar{g}_*\left(w\partial_{u|v} + w\partial_{v|u}\right) = [\ominus_2]$

Induced DGA Structure

- $\bar{g} : Hom\left(H^{\otimes 2}, H\right) \to Hom\left(H^{\otimes 2}, A\right)$ is a quasi-isomorphism

	2	4	5	7	9	10	12	14
H	u		v	w				
$H \otimes H$		$u\|u$		$u\|v,\ v\|u$	$u\|w,\ w\|u$	$v\|v$	$v\|w,\ w\|v$	$w\|w$

- Evaluate $\bar{g}$ on basis $\left\{ u|v \overset{w\partial_{u|v}}{\mapsto} w,\ v|u \overset{w\partial_{v|u}}{\mapsto} w \right\} \subset Hom^0\left(H^{\otimes 2}, H\right)$ and evaluate $\ominus_2 = \mu\left(g \otimes g\right)$ on basis $\{u|v, v|u\}$ in degree 7

$$u|v \overset{w\partial_{u|v}}{\mapsto} w \overset{g}{\mapsto} a_2\left(a_2 a_3 + a_3 a_2\right) = \ominus_2\left(u|v\right)$$
$$v|u \overset{w\partial_{v|u}}{\mapsto} w \overset{g}{\mapsto} \left(a_2 a_3 + a_3 a_2\right) a_2 = \ominus_2\left(v|u\right)$$

- Thinking of $w\partial_{u|v} + w\partial_{v|u}$ as a class, $\bar{g}_*\left(w\partial_{u|v} + w\partial_{v|u}\right) = \left[\ominus_2\right]$

- Define $\mu_H = \left(\bar{g}_*\right)^{-1}\left[\ominus_2\right] = w\partial_{u|v} + w\partial_{v|u}$; then $uv = vu = w$

Extending g to an $A(2)$-map

- The non-trivial values of $\mu(g \otimes g) - g\mu_H$ are

$$a_2^2 \partial_{u|u}, \quad \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{u|w}, \text{ and } \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{w|u}$$

Extending g to an $A(2)$-map

- The non-trivial values of $\mu(g \otimes g) - g\mu_H$ are

$$a_2^2 \partial_{u|u}, \quad \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{u|w}, \text{ and } \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{w|u}$$

- These cocycles cobound

$$\begin{aligned}
\nabla\left(a_3 \partial_{u|u}\right) &= da_3 \partial_{u|u} = a_2^2 \partial_{u|u} \\
\nabla\left(a_3 a_2 a_3 \partial_{u|w}\right) &= d\left(a_3 a_2 a_3\right) \partial_{u|w} = \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{u|w} \\
\nabla\left(a_3 a_2 a_3 \partial_{w|u}\right) &= d\left(a_3 a_2 a_3\right) \partial_{w|u} = \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{w|u}
\end{aligned}$$

Extending g to an $A(2)$-map

- The non-trivial values of $\mu(g \otimes g) - g\mu_H$ are

$$a_2^2 \partial_{u|u}, \quad \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{u|w}, \text{ and } \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{w|u}$$

- These cocycles cobound

$$\begin{aligned}
\nabla\left(a_3 \partial_{u|u}\right) &= da_3 \partial_{u|u} = a_2^2 \partial_{u|u} \\
\nabla\left(a_3 a_2 a_3 \partial_{u|w}\right) &= d\left(a_3 a_2 a_3\right) \partial_{u|w} = \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{u|w} \\
\nabla\left(a_3 a_2 a_3 \partial_{w|u}\right) &= d\left(a_3 a_2 a_3\right) \partial_{w|u} = \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{w|u}
\end{aligned}$$

- Thus we define

$$g_2 = a_3 \partial_{u|u} + a_3 a_2 a_3 \left(\partial_{u|w} + \partial_{w|u}\right)$$

Extending g to an $A(2)$-map

- The non-trivial values of $\mu(g \otimes g) - g\mu_H$ are

$$a_2^2 \partial_{u|u}, \quad \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{u|w}, \text{ and } \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{w|u}$$

- These cocycles cobound

$$\begin{aligned} \nabla\left(a_3 \partial_{u|u}\right) &= da_3 \partial_{u|u} = a_2^2 \partial_{u|u} \\ \nabla\left(a_3 a_2 a_3 \partial_{u|w}\right) &= d\left(a_3 a_2 a_3\right) \partial_{u|w} = \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{u|w} \\ \nabla\left(a_3 a_2 a_3 \partial_{w|u}\right) &= d\left(a_3 a_2 a_3\right) \partial_{w|u} = \left(a_2^3 a_3 + a_3 a_2^3\right) \partial_{w|u} \end{aligned}$$

- Thus we define

$$g_2 = a_3 \partial_{u|u} + a_3 a_2 a_3 \left(\partial_{u|w} + \partial_{w|u}\right)$$

- Then $\nabla g_2 = \mu(g \otimes g) - g\mu_H$ and $g + g_2$ is an A_2-map

Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\ominus_3 = \mu\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_H \otimes 1 - 1 \otimes \mu_H\right)$

Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\ominus_3 = \mu\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_H \otimes 1 - 1 \otimes \mu_H\right)$
- $\bar{g} : Hom\left(H^{\otimes 3}, H\right) \to Hom\left(H^{\otimes 3}, A\right)$ is a quasi-isomorphism

	2	5	6	7	$\cdots$
H	u	v		w	
$H \otimes H \otimes H$			$u\|u\|u$		$\cdots$

Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\ominus_3 = \mu\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_H \otimes 1 - 1 \otimes \mu_H\right)$
- $\bar{g} : Hom\left(H^{\otimes 3}, H\right) \to Hom\left(H^{\otimes 3}, A\right)$ is a quasi-isomorphism

	2	5	6	7	$\cdots$
H	u	v		w	
$H \otimes H \otimes H$			$u\|u\|u$		$\cdots$

- Evaluate $\bar{g}$ on the basis $\left\{ u|u|u \overset{v\partial_{u|u|u}}{\longmapsto} v \right\}$ for $Hom^{-1}\left(H^{\otimes 3}, H\right)$ and evaluate $\ominus_3$ on the basis $\{u|u|u\}$ in degree 6

$$u|u|u \overset{v\partial_{u|u|u}}{\longmapsto} v \overset{g}{\longmapsto} a_2 a_3 + a_3 a_2 = \ominus_3\left(u|u|u\right)$$

Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\ominus_3 = \mu\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_H \otimes 1 - 1 \otimes \mu_H\right)$
- $\bar{g} : Hom\left(H^{\otimes 3}, H\right) \to Hom\left(H^{\otimes 3}, A\right)$ is a quasi-isomorphism

	2	5	6	7	$\cdots$
H	u	v		w	
$H \otimes H \otimes H$			$u\vert u\vert u$		$\cdots$

- Evaluate $\bar{g}$ on the basis $\left\{ u|u|u \overset{v\partial_{u|u|u}}{\mapsto} v \right\}$ for $Hom^{-1}\left(H^{\otimes 3}, H\right)$ and evaluate $\ominus_3$ on the basis $\{u|u|u\}$ in degree 6

$$u|u|u \overset{v\partial_{u|u|u}}{\mapsto} v \overset{g}{\mapsto} a_2 a_3 + a_3 a_2 = \ominus_3\left(u|u|u\right)$$

- Thinking of $v\partial_{u|u|u}$ as a class: $\bar{g}_*\left(v\partial_{u|u|u}\right) = [\ominus_3]$

Induced $A(3)$-structure

- $\mu^3 = 0$ implies $\ominus_3 = \mu(g_2 \otimes g - g \otimes g_2) + g_2(\mu_H \otimes 1 - 1 \otimes \mu_H)$
- $\bar{g} : Hom\left(H^{\otimes 3}, H\right) \to Hom\left(H^{\otimes 3}, A\right)$ is a quasi-isomorphism

	2	5	6	7	$\cdots$
H	u	v		w	
$H \otimes H \otimes H$			$u\|u\|u$		$\cdots$

- Evaluate $\bar{g}$ on the basis $\left\{ u|u|u \overset{v\partial_{u|u|u}}{\longmapsto} v \right\}$ for $Hom^{-1}\left(H^{\otimes 3}, H\right)$ and evaluate $\ominus_3$ on the basis $\{u|u|u\}$ in degree 6

$$u|u|u \overset{v\partial_{u|u|u}}{\longmapsto} v \overset{g}{\longmapsto} a_2 a_3 + a_3 a_2 = \ominus_3(u|u|u)$$

- Thinking of $v\partial_{u|u|u}$ as a class: $\bar{g}_*\left(v\partial_{u|u|u}\right) = [\ominus_3]$
- Define $\mu_H^3 = (\bar{g}_*)^{-1}[\ominus_3] = v\partial_{u|u|u}$; then $\mu_H^3(u|u|u) = v$

Induced Higher Order Structure

- $\ominus_3 - g\mu_H^3 \equiv 0$

Induced Higher Order Structure

- $\ominus_3 - g\mu_H^3 \equiv 0$
- Define $g_n = 0$ and $\mu_H^n = 0$ for all $n \geq 4$

Induced Higher Order Structure

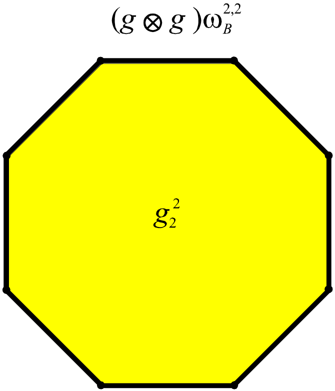
- $\ominus_3 - g\mu_H^3 \equiv 0$
- Define $g_n = 0$ and $\mu_H^n = 0$ for all $n \geq 4$
- Then $\left(H, \mu_H, \mu_H^3\right)$ is an A_∞-algebra and $G = g + g_2$ is A_∞-map

Induced Higher Order Structure

- $\ominus_3 - g\mu_H^3 \equiv 0$
- Define $g_n = 0$ and $\mu_H^n = 0$ for all $n \geq 4$
- Then (H, μ_H, μ_H^3) is an A_∞-algebra and $G = g + g_2$ is A_∞-map
- Theorem 1 extends immediately to A_∞-bialgebras

Generalized Multiplihedra

Transfer is controlled by generalized multiplihedra $\{JJ_{m,n}\}_{m,n\geq 1}$ of which $JJ_{n,1} = JJ_{1,n} = J_n$



The Generalized Multiplihedron $JJ_{2,2}$

Transfer Problem 2

- The problem is exactly the same as before

Transfer Problem 2

- The problem is exactly the same as before
- Identify $g_{n,m}$ with the $(m+n-1)$-dimensional polyhedron $J_{n,m}$

Transfer Problem 2

- The problem is exactly the same as before
- Identify $g_{n,m}$ with the $(m+n-1)$-dimensional polyhedron $J_{n,m}$
- Let $Q^{m+n-2} \subset JJ_{n,m}$ denote the cell identified with $g^{\otimes n}\omega_B^{n,m}$

Transfer Problem 2

- The problem is exactly the same as before
- Identify $g_{n,m}$ with the $(m+n-1)$-dimensional polyhedron $J_{n,m}$
- Let $Q^{m+n-2} \subset JJ_{n,m}$ denote the cell identified with $g^{\otimes n}\omega_B^{n,m}$
- $\ominus_m^n \in Hom^{m+n-2}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with $\partial JJ_{n,m} - \operatorname{int} Q^{m+n-2}$

Transfer Problem 2

- The problem is exactly the same as before
- Identify $g_{n,m}$ with the $(m+n-1)$-dimensional polyhedron $J_{n,m}$
- Let $Q^{m+n-2} \subset JJ_{n,m}$ denote the cell identified with $g^{\otimes n}\omega_B^{n,m}$
- $\ominus_m^n \in Hom^{m+n-2}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with $\partial JJ_{n,m} - \operatorname{int} Q^{m+n-2}$
- $\nabla \ominus_m^n \in Hom^{m+n-3}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with ∂Q^{m+n-2}

Transfer Problem 2

- The problem is exactly the same as before
- Identify $g_{n,m}$ with the $(m+n-1)$-dimensional polyhedron $J_{n,m}$
- Let $Q^{m+n-2} \subset JJ_{n,m}$ denote the cell identified with $g^{\otimes n}\omega_B^{n,m}$
- $\ominus_m^n \in Hom^{m+n-2}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with $\partial JJ_{n,m} - \operatorname{int} Q^{m+n-2}$
- $\nabla\ominus_m^n \in Hom^{m+n-3}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with ∂Q^{m+n-2}
- **Transfer Problem 2:** *Let A be an A_∞-bialgebra, let B be a* DGM,*and let $g : B \to A$ be a chain map. Given $\left\{\omega_B^{j,i},\ g_i^j\right\}_{1\leq i+j<k}$ construct g_m^n* and $\omega_B^{n,m}$ for each (m,n) with $m+n=k$ *so that*

$$\nabla g_m^n = \ominus_m^n - g^{\otimes n}\omega_B^{n,m}$$

Second Transfer Theorem

- A chain map g induces a cochain map

$$\tilde{g} : \left(Hom\left(B^{\otimes m}, B^{\otimes n} \right); \nabla_B \right) \rightarrow Hom\left(B^{\otimes m}, A^{\otimes n}; \nabla \right)$$

given by $\tilde{g}(u) = g^{\otimes n} u$

Second Transfer Theorem

- A chain map g induces a cochain map

$$\tilde{g} : \left(Hom\left(B^{\otimes m}, B^{\otimes n}\right); \nabla_B\right) \to Hom\left(B^{\otimes m}, A^{\otimes n}; \nabla\right)$$

  given by $\tilde{g}\left(u\right) = g^{\otimes n} u$

- Transfer Problem 2 has a solution whenever $\tilde{g}$ is a quasi-isomorphism

Second Transfer Theorem

- A chain map g induces a cochain map

$$\tilde{g} : \left(Hom\left(B^{\otimes m}, B^{\otimes n}\right); \nabla_B \right) \to Hom\left(B^{\otimes m}, A^{\otimes n}; \nabla\right)$$

given by $\tilde{g}(u) = g^{\otimes n} u$

- Transfer Problem 2 has a solution whenever $\tilde{g}$ is a quasi-isomorphism

- **Theorem 2.** *If $\tilde{g}$ is a quasi-isomorphism, then*

  (i) g transfers the A_∞-bialgebra structure from A to B; the induced structure on B is unique up to automorphism

  (ii) g extends to a map $G : B \Rightarrow A$ of A_∞-bialgebras

- **Proposition 2.** *Let A* be *an A_∞-bialgebra A over a field* $\mathbf{k}$, and choose a cycle-selecting homomorphism $g : H \longrightarrow A$*. Then $\tilde{g}$ is a quasi-isomorphism.*

Application: Homology of Loop Spaces

- **Proposition 2.** *Let* A be *an* A_∞*-bialgebra* A *over a field* $\mathbf{k}$, and choose a cycle-selecting homomorphism $g : H \to A$. *Then* $\tilde{g}$ *is a quasi-isomorphism.*

- **Corollary 2.** *If* A *is an* A_∞*-bialgebra over a field* $\mathbf{k}$, *there is an induced* A_∞*-bialgebra structure on* $H\left(A;\mathbf{k}\right)$.

Application: Homology of Loop Spaces

- **Proposition 2.** *Let A* be *an A_∞-bialgebra A over a field* $\mathbf{k}$, and choose a cycle-selecting homomorphism $g : H \longrightarrow A$. *Then $\tilde{g}$ is a quasi-isomorphism.*

- **Corollary 2.** *If A is an A_∞-bialgebra over a field $\mathbf{k}$, there is an induced A_∞-bialgebra structure on $H\left(A;\mathbf{k}\right)$.*

- **Example 2.** *Let X be a space. There is an induced A_∞-bialgebra structure on $H_*\left(\Omega X;\mathbf{k}\right)$*

Rational Cohomology of Loop Spaces

- **Example 3.** Let X be a simply connected space

Rational Cohomology of Loop Spaces

- **Example 3.** Let X be a simply connected space
- $H = H^*(\Omega X; \mathbb{Q})$ is an A_∞-coalgebra with operations

$$\left\{ \Delta^n : H \to H^{\otimes n} \right\}_{n \geq 2}$$

Rational Cohomology of Loop Spaces

- **Example 3.** Let X be a simply connected space
- $H = H^*(\Omega X; \mathbb{Q})$ is an A_∞-coalgebra with operations

$$\left\{\Delta^n : H \to H^{\otimes n}\right\}_{n \geq 2}$$

- Compatibility of the cup product μ with $\Delta = \Delta^2$ is expressed by the classical Hopf relation

$$\Delta\mu = (\mu \otimes \mu)\,\sigma_{2,2}\,(\Delta \otimes \Delta)$$

Rational Cohomology of Loop Spaces

- **Example 3.** Let X be a simply connected space
- $H = H^*(\Omega X; \mathbb{Q})$ is an A_∞-coalgebra with operations

$$\left\{\Delta^n : H \to H^{\otimes n}\right\}_{n\geq 2}$$

- Compatibility of the cup product μ with $\Delta = \Delta^2$ is expressed by the classical Hopf relation

$$\Delta\mu = (\mu\otimes\mu)\,\sigma_{2,2}\,(\Delta\otimes\Delta)$$

- Compatibility of μ with Δ^3 is expressed by the relation

$$\Delta^3\mu = \mu^{\otimes 3}\sigma_{3,2}\left[(\Delta\otimes 1)\,\Delta\otimes\Delta^3 + \Delta^3\otimes(1\otimes\Delta)\,\Delta\right],$$

where $\sigma_{p,q} : \left(H^{\otimes p}\right)^{\otimes q} \to \left(H^{\otimes q}\right)^{\otimes p}$ permutes tensor factors

Rational Cohomology of Loop Spaces (continued)

- Compatibility of μ with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$$

Rational Cohomology of Loop Spaces (continued)

- Compatibility of μ with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$$

- Let e^{n-2} denote the top dimensional cell of K_n

Rational Cohomology of Loop Spaces (continued)

- Compatibility of μ with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \to C_*(K) \otimes C_*(K)$$

- Let e^{n-2} denote the top dimensional cell of K_n
- Represent the A_∞-coalgebra H as an algebra over the A_∞-operad

$$\xi : C_*(K) \to Hom\left(H, H^{\otimes n}\right)$$

Rational Cohomology of Loop Spaces (continued)

- Compatibility of μ with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

$$\Delta_K : C_*(K) \rightarrow C_*(K) \otimes C_*(K)$$

- Let e^{n-2} denote the top dimensional cell of K_n
- Represent the A_∞-coalgebra H as an algebra over the A_∞-operad

$$\xi : C_*(K) \rightarrow Hom\left(H, H^{\otimes n}\right)$$

- Then $\xi\left(e^{n-2}\right) = \Delta^n$ and

$$\Delta^n \mu = \mu^{\otimes n} \sigma_{n,2}\left[(\xi \otimes \xi)\, \Delta_K\left(e^{n-2}\right)\right]$$

The End

Thank you!