Some Naturally Occurring Examples of A_∞-Bialgebras

Presented by

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Overview

In her 2006 thesis, A. Berciano applied perturbation methods of T. Kadeishvili and others to reveal the A_∞-coalgebra structure $\{\Delta_2, \Delta_p\}$ on tensor factors $E \otimes \Gamma \subset H = H_*(\mathbb{Z}, n; \mathbb{Z}_p)$.

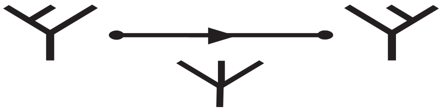
Since H is a Hopf algebra, Δ_2 is compatible with the Pontryagin product μ as a map of algebras, and it is natural to ask whether Δ_p is in some sense compatible with μ as well.

Indeed, Δ_p is compatible with μ as a “higher derivation.” Thus $E \otimes \Gamma$ is a naturally occurring example of an A_∞-bialgebra whose simple internal structure is easily understood.

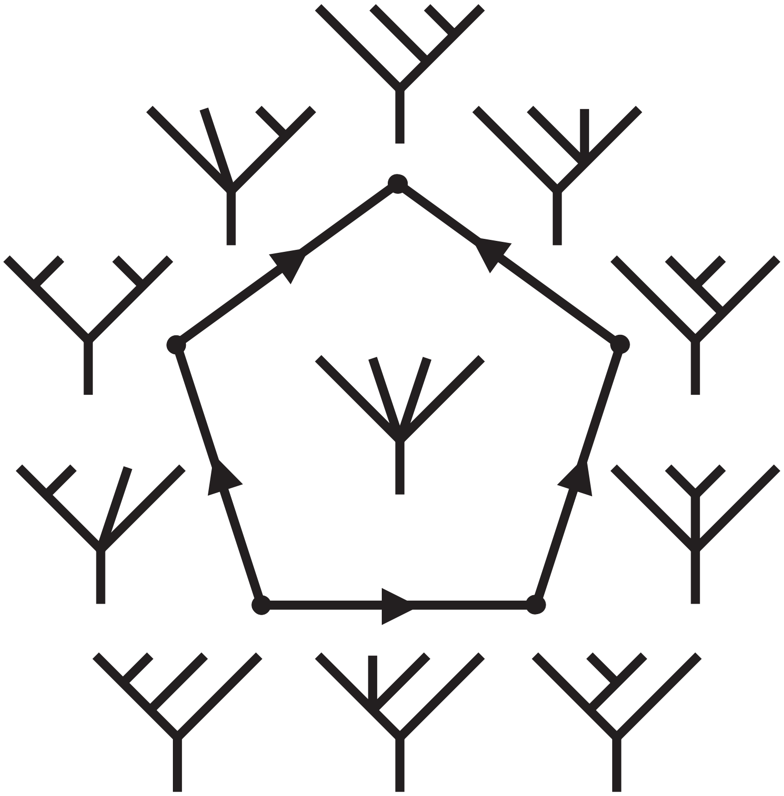
Stasheff's Associahedra and the $\mathcal{A}_\infty$-operad

- K_2:

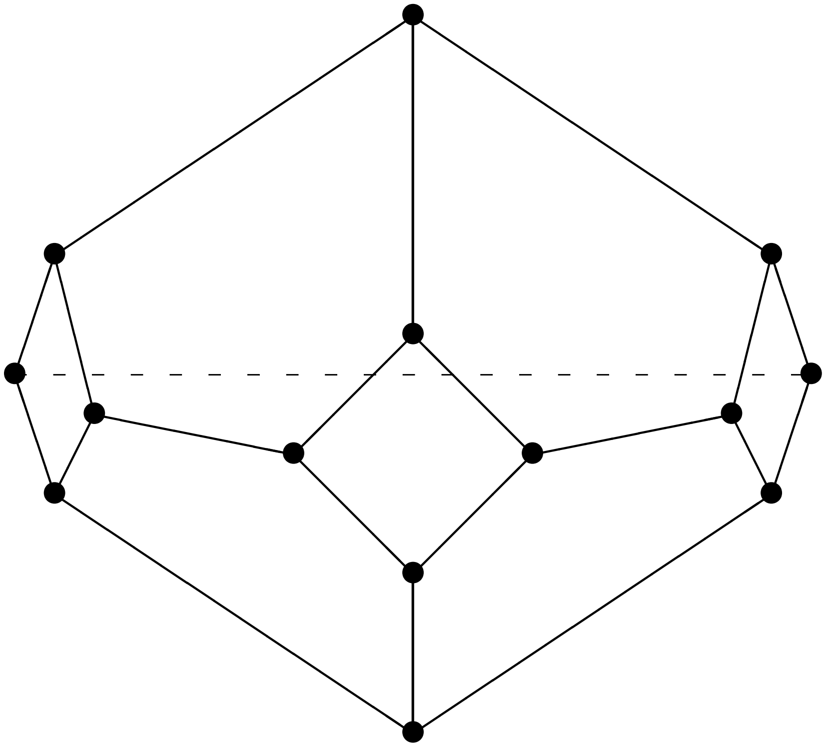
- K_3:



- K_4:

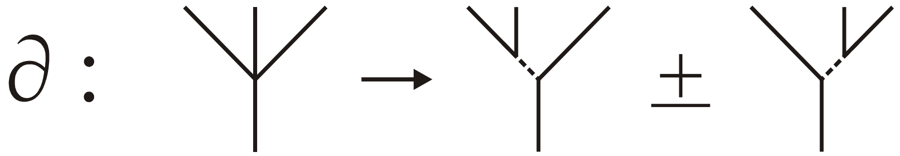


- K_5 is 3-dimensional polytope:



- K_n is an $(n-2)$-dimensional polytope
- $K = \sqcup K_n$
- Cellular chains $C_*(K)$ represent the $\mathcal{A}_\infty$ operad

- Given a tree $T \in C_*(K)$ with k (internal) nodes, $\partial T = \sum \pm T_i$, where the T_i's have $k+1$ nodes and are obtained from T by inserting an internal branch at each node of T with valance ≥ 3 in all possible ways



- $A^{\otimes n}$ is a DGM with differential

$$d^{\otimes} = \sum_{i=0}^{n-1} 1^{\otimes i} \otimes d \otimes 1^{\otimes n-i-1}$$

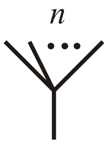
- $Hom^*(A, B)$ is a DGM with differential

$$\delta(f) = d_B f - (-1)^{|f|} f d_A$$

- A chain map $\xi : C_*(K_n) \rightarrow Hom^*(A, A^{\otimes n})$ defines an A_∞-*coalgebra structure* on A

Definition: An A_∞-*coalgebra* is a DGM A together with a chain map

$$\xi : C_*\left(K_n\right) \to Hom^*\left(A, A^{\otimes n}\right)$$

- Operation $\Delta_n := \xi$ ()

- Structure relations are given by

$$\begin{tikzcd} C_*\left(K_n\right) \arrow[r, "\xi"] \arrow[d, "\partial"'] & Hom^*\left(A, A^{\otimes n}\right) \arrow[d, "\delta"] \\ C_{*-1}\left(K_n\right) \arrow[r, "\xi"'] & Hom^{*-1}\left(A, A^{\otimes n}\right) \end{tikzcd}$$

- $n = 3 : \quad \delta\left(\Delta_3\right) = \xi$ ( $-$ )

$$\left(d \otimes 1^{\otimes 2} + 1 \otimes d \otimes 1 + 1^{\otimes 2} \otimes d\right) \Delta_3 + \Delta_3 d$$

$$= (1 \otimes \Delta)\,\Delta - (\Delta \otimes 1)\,\Delta$$

A_∞-coalgebras and Tilde-cobar Construction

Given an *arbitrary* family of maps

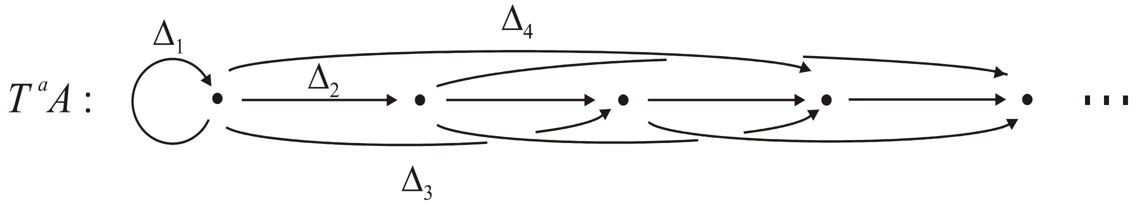
$$\left\{ \Delta_n \in Hom^{n-2}\left(A, A^{\otimes n}\right) \right\}_{n \geq 1}$$

extend $\omega = \Delta_1 + \Delta_2 + \cdots : A \to T^a A$

as a derivation $\overline{\omega}$ of $T^a A$. Then (up to sign)

$$\begin{aligned} \overline{\omega} \; = \; & \Delta_1 + (\Delta_1 \otimes 1 + 1 \otimes \Delta_1) + \cdots \\ & + \Delta_2 + (\Delta_2 \otimes 1 + 1 \otimes \Delta_2) + \cdots \\ & + \Delta_3 + (\Delta_3 \otimes 1 + 1 \otimes \Delta_3) + \cdots \\ & \\ & \in Der\,(T^a A) \end{aligned}$$

- $(T^a A, \overline{\omega})$ is the *tilde-cobar construction* on A



- $(A, \Delta_n)_{n \geq 1}$ is an A_∞-*coalgebra* if $\overline{\omega} \circ \overline{\omega} = 0$

- Structure Relations:

For simplicity, let $d = \Delta_1$ and $\Delta = \Delta_2$. Then

$$0 = \overline{\omega} \circ \overline{\omega} = [dd] + [(d \otimes 1 + 1 \otimes d)\,\Delta - \Delta d] +$$

$$\Big[\big(d \otimes 1^{\otimes 2} + 1 \otimes d \otimes 1 + 1^{\otimes 2} \otimes d\big)\,\Delta_3 + \Delta_3 d$$
$$- (1 \otimes \Delta - \Delta \otimes 1)\,\Delta\Big] +$$

$$\Big[\big(d \otimes 1^{\otimes 3} + \cdots + 1^{\otimes 3} \otimes d\big)\,\Delta_4 - \Delta_4 d$$
$$+ \big(\Delta \otimes 1^{\otimes 2} - 1 \otimes \Delta \otimes 1 + 1^{\otimes 2} \otimes \Delta\big)\,\Delta_3$$
$$- (\Delta_3 \otimes 1 + 1 \otimes \Delta_3)\,\Delta\Big] + \cdots$$

and by homogeneity we have

1. $dd = 0$
2. $(d \otimes 1 + 1 \otimes d)\,\Delta - \Delta d = 0$
3. $d^{\otimes}\Delta_3 + \Delta_3 d = (1 \otimes \Delta - \Delta \otimes 1)\,\Delta$
4. $d^{\otimes}\Delta_4 - \Delta_4 d = (\Delta_3 \otimes 1 + 1 \otimes \Delta_3)\,\Delta$
   $- \big(\Delta \otimes 1^{\otimes 2} - 1 \otimes \Delta \otimes 1 + 1^{\otimes 2} \otimes \Delta\big)\,\Delta_3$

Simple A_∞-coalgebras

Definition: A *simple A_∞-coalgebra* is an A_∞-coalgebra of form (A, d, Δ, Δ_n).

Thus

- (A, d, Δ) is a DG coalgebra
- If $n = 3$, Δ is homotopy coassociative
  $d^{\otimes}\Delta_3 + \Delta_3 d = (1 \otimes \Delta)\,\Delta - (\Delta \otimes 1)\,\Delta$
- If $n > 3$, Δ is strictly coassociative

$$(1 \otimes \Delta)\,\Delta = (\Delta \otimes 1)\,\Delta \text{ and}$$

$$d^{\otimes}\Delta_n - (-1)^n\,\Delta_n d = 0$$

- $\displaystyle\sum_{i=0}^{n-1} (-1)^i \left(1^{\otimes i} \otimes \Delta \otimes 1^{\otimes n-i-1}\right)\Delta_n$
  $\displaystyle= (1 \otimes \Delta_n - (-1)^n\,\Delta_n \otimes 1)\,\Delta$
- $\displaystyle\sum_{i=0}^{n-1} (-1)^{i(n+1)} \left(1^{\otimes i} \otimes \Delta_n \otimes 1^{\otimes n-i-1}\right)\Delta_n = 0$

Theorem (Eilenberg, Mac Lane) *For odd primes* p,

$$H_*(\mathbb{Z}, n; \mathbb{Z}_p) \approx$$

$$\bigotimes_{i \geq 0} E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2)$$

Theorem (Berciano) *For odd primes* p *and* $i \geq 0$,

$$E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2)$$

is a simple A_∞*-coalgebra with* v_i *and* w_i *primitive and* Δ_p *defined as follows: Set* $\gamma_j = \gamma_j(w_i)$; *then on a monomial* x,

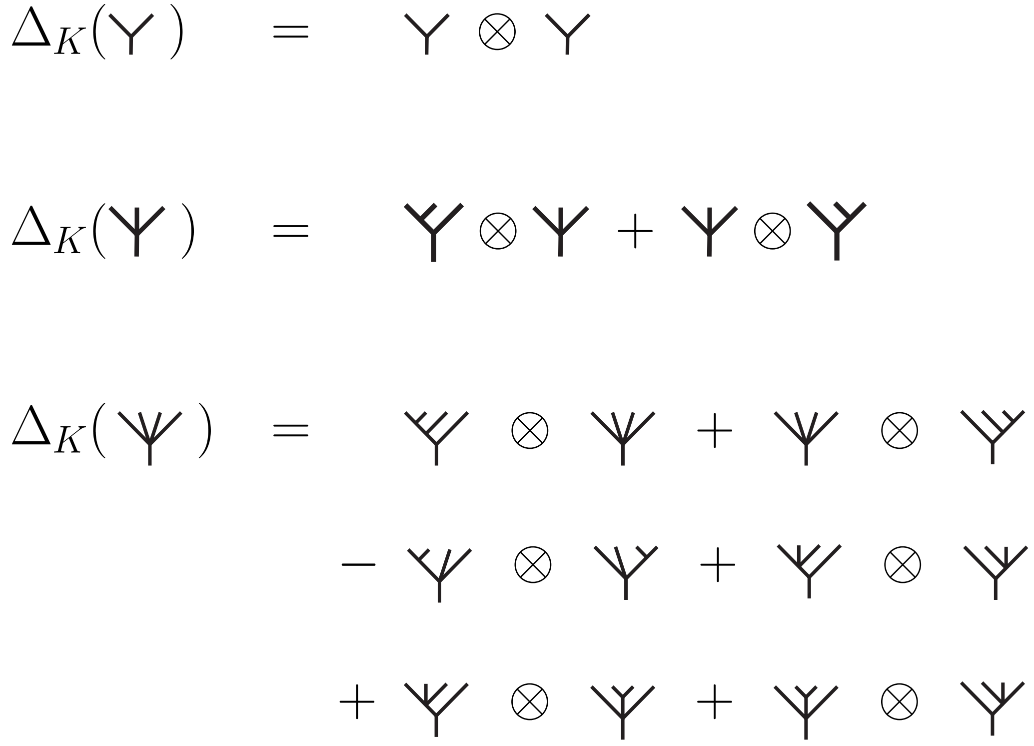
$$\Delta_p(x) = \begin{cases} \displaystyle\sum_{k_1 + \cdots + k_p = j-1} v_i \gamma_{k_1} | \cdots | v_i \gamma_{k_p}, & \text{if } x = \gamma_j \\ 0, & \text{otherwise.} \end{cases}$$

Question: Δ is compatible with the multiplication μ in the sense that Δ is an algebra map. In what sense is Δ_p compatible with μ?

Δ_K-Derivations (Saneblidze, U)

- Diagonal on Associahedra: Define

$\Delta_K : C_*(K_n) \to C_*(K_n) \otimes C_*(K_n)$ by



$\vdots$

Let (H,ξ) be an A_∞-coalgebra with associative multiplication $\mu : H \otimes H \to H$.

- If Δ is an algebra map, H is a bialgebra

$$\begin{aligned}\Delta\mu &= \mu^{\otimes 2}\sigma_{2,2}(\Delta\otimes\Delta)\\ &= \mu^{\otimes 2}\sigma_{2,2}\left[(\xi\otimes\xi)\,\Delta_K(\Upsilon)\right]\end{aligned}$$

- Δ is a Δ_K-*derivation with respect to the family* $\mathfrak{F}_2 = \varnothing$

- $\mathfrak{F}_3 = \{f = (\Delta\otimes 1)\,\Delta,\ g = (1\otimes\Delta)\,\Delta\}$

$$\leftrightarrow \{\text{vertices of } K_3 = I\}$$

is a Δ_K-*compatible family on the* 0-*skeleton* $K_3^{(0)}$

- If Δ_3 is a (f,g)-derivation of μ,

$$\begin{aligned}\Delta_3\mu &= \mu^{\otimes 3}\sigma_{3,2}\left[f\otimes\Delta_3+\Delta_3\otimes g\right]\\ &= \mu^{\otimes 3}\sigma_{3,2}\left[(\xi\otimes\xi)\,\Delta_K\left(\right.\right.\end{aligned}$$

$$\left.\left.\right)\right]$$

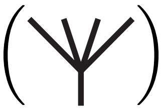
- Δ_3 is a Δ_K-*derivation with respect to* $\mathfrak{F}_3$

- $\mathfrak{F}_4=\{$compositions involving $\Delta_2,\Delta_3\}$

  $\leftrightarrow\{$vertices and edges of $K_4\}$

  is a Δ_K-*compatible family on* $K_4^{(1)}$

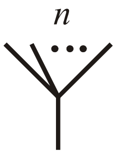
- If Δ_4 is a Δ_K-*derivation with respect to* $\mathfrak{F}_4$,

$$\Delta_4\mu=\mu^{\otimes 4}\sigma_{4,2}\left[(\xi\otimes\xi)\,\Delta_K\left(\right.\right.$$

$$\left.\left.\right)\right]$$

Inductively:

$\mathfrak{F}_n=\{$compositions involving $\Delta_2,\ldots,\Delta_{n-1}\}$

Δ_n is a Δ_K-*derivation with respect to* $\mathfrak{F}_n$ if

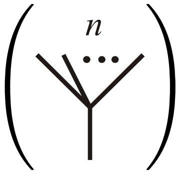
$$\Delta_n\mu=\mu^{\otimes n}\sigma_{n,2}\left[(\xi\otimes\xi)\,\Delta_K\left(\right.\right.$$


$$\left.\left.\right)\right]$$

Note: When H is simple, Δ is coassociative $(d=0)$ and $\Delta_i \neq 0$ iff $i = 2, n$. Let

$$f_n \;=\; \left(\Delta \otimes 1^{\otimes n-2}\right) \cdots (\Delta \otimes 1)\,\Delta$$

Then $\mathfrak{F}_n = \{0, f_n\}$ is a Δ_K-compatible family in which f_n is associated with each vertex of K_n and 0 is associated with all other faces of $K_n^{(n-3)}$. Thus

$(\xi \otimes \xi)\,\Delta_K$  $= f_n \otimes \Delta_n + \Delta_n \otimes f_n$

so that Δ_n *is a* Δ_K*-derivation wrt to* $\mathfrak{F}_n$ *if*

$$\Delta_n \mu \;=\; \mu^{\otimes n} \sigma_{n,2} \left(f_n \otimes \Delta_n + \Delta_n \otimes f_n\right)$$

Hopf A_∞-coalgebras

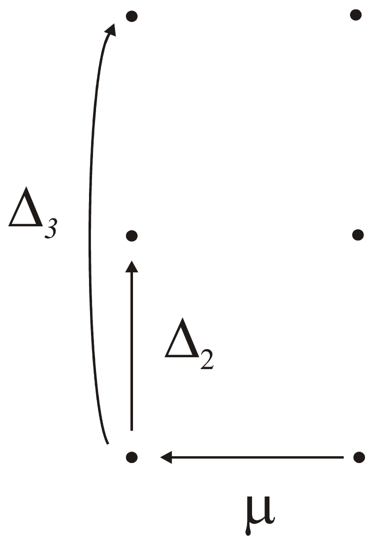
Structure relations in a general A_∞-bialgebra arise from the homogeneous components of a square-zero differential on some universal complex in the same way that the structure relations in a A_∞-coalgebra arise from the homogeneous components of the differential on the tilde-cobar construction.

Given arbitrary maps $\Delta : H \to H \otimes H$, $\Delta_n : H \to H^{\otimes n}$ and $\mu : H \otimes H \to H$ consider $\omega = \mu + \Delta + \Delta_n$. Extend

- $\Delta + \Delta_n$ as a derivation of $T^a H$;
- Δ as an algebra map $T^a H \to T^a \left(H^{\otimes 2}\right)$;
- Δ_n as a Δ_K-derivation wrt $\mathfrak{F}_n = \{f_n, 0\}$;
- μ as a coalgebra map $T^c \left(H^{\otimes 2}\right) \to T^c H$.

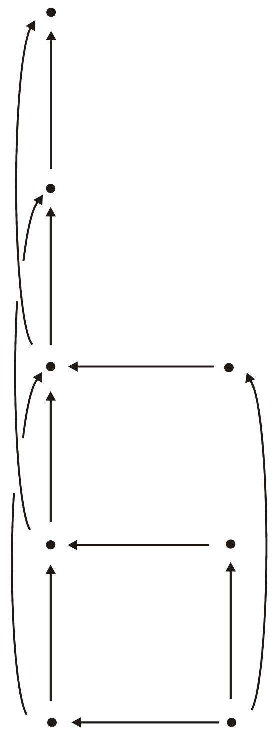
The *biderivative of* ω, denoted here by $\overline{\omega}$, is the sum of these (unique) extensions.

Identify $\left(H^{\otimes p}\right)^{\otimes q} \approx \left(H^{\otimes q}\right)^{\otimes p}$ with $(p, q) \in \mathbb{N}^2$ and picture initial maps as arrows along the axes:



Initial maps with $n = 3$.

Consider components of the biderivative that form a 2-step chain from the x to the y axis:



Components of $\overline{\varpi}$ when $n = 3$.

- Components of $\overline{\Delta + \Delta_3} : T^a H \to T^a H$ are vertical arrows along the y-axis
- Component $f_3 \otimes \Delta_3 + \Delta_3 \otimes f_3$ of $\overline{\Delta_3}$ is the vertical arrow $(2,1) \to (2,3)$

For $g : (H^{\otimes r})^{\otimes s} \to (H^{\otimes p})^{\otimes q}$ and

$f : (H^{\otimes q})^{\otimes p} \to (H^{\otimes t})^{\otimes u}$ define

$$f \circledcirc g = f \circ \sigma_{p,q} \circ g,$$

where $\sigma_{p,q} : (H^{\otimes p})^{\otimes q} \to (H^{\otimes q})^{\otimes p}$ is the standard permutation of tensor factors.

Definition: A *simple Hopf A_∞-coalgebra* is a DGM H together with maps $\Delta : H \to H \otimes H$, $\Delta_n : H \to H^{\otimes n}$ and $\mu : H \otimes H \to H$ such that $\omega = \Delta + \Delta_n + \mu$ implies $\overline{\omega} \circledcirc \overline{\omega} = 0$.

Equivalently,

1. (H, Δ, Δ_n) is a simple A_∞-coalgebra
2. (H, Δ, μ) is a (biassociative) Hopf algebra
3. $\Delta_n \mu = \mu^{\otimes n} \sigma_{n,2} \left(f \otimes \Delta_n + \Delta_n \otimes f \right).$

Theorem (B-U) *For odd primes p and $i \geq 0$,*

$$E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2)$$

is a simple Hopf A_∞-coalgebra.

Proof: The fact that Δ_p is a Δ-derivation wrt $\mathfrak{F}_p = \{0, f_p\}$ follows from the identity below.

Lemma (B-U) For all $(p_1, \ldots, p_n) \in \mathbb{Z}_p^n$,

$$\binom{p_1 + \cdots + p_n + 1}{k} =$$

$$\sum_{q_1+\cdots+q_n=k-1} \binom{p_1}{q_1} \cdots \binom{p_n}{q_n}$$

$$+ \sum_{r_1+\cdots+r_n=k} \binom{p_1}{r_1} \cdots \binom{p_n}{r_n}.$$

Proof: Apply *Vandermonde's Identity*:

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}.$$

Remark: Extending the Hopf A_∞-coalgebra structure on each factor $E \otimes \Gamma$ of $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$ to a global A_∞-bialgebra structure requires tensor products of A_∞-bialgebras. We expect such an extension to agree with the A_∞-bialgebra structure on the loop space homology of $K(\mathbb{Z}, n)$.