A_∞-Bialgebras and the Mod 3 Homology of K(Z,n)

Presented by

Ron Umble

Millersville University of Pennsylvania
ron.umble@millersville.edu

Joint work with

Ainhoa Berciano and Samson Saneblidze

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Abstract

A. Berciano applied perturbation methods of T. Kadeishvili and others to define an A_∞-coalgebra structure $\left\{\Delta_i : H \to H^{\otimes i}\right\}_{i\geq 2}$ on $H = H_*(\mathbb{Z}, n; \mathbb{Z}_p)$. Since H is a Hopf algebra, Δ_2 is compatible with the Pontryagin product μ as a map of algebras, and it is natural to ask whether the higher order Δ_i's are in some sense compatible with μ as well.

Indeed, let $f = (\Delta_2 \otimes 1)\,\Delta_2$; when $p = 3$, there is a tensor factor of H of form $E \otimes \Gamma$ on which there are exactly two non-vanishing A_∞-coalgebra operations Δ_2 and Δ_3. We prove that Δ_3 is compatible with μ as an (f, f)-derivation. Thus $E \otimes \Gamma$ is a naturally occurring example of a Hopf A_∞-coalgebra.

Furthermore, H admits operations Δ_i, for all $i \geq 2$. Since H is the homology of a DG Hopf algebra, all A_∞-bialgebra structure relations involving Δ_i and μ are satisfied. But whether or not Δ_i is compatible with μ as some "higher derivation" is an open question.

General A_∞-bialgebras

- Classical constructions:

  a. Chain maps

$$\mathcal{A}ss \ \rightarrow \ Hom\left(A^{\otimes n}, A\right)$$

$$\mathcal{A}_\infty \rightarrow Hom\left(A^{\otimes n}, A\right)$$

  in the category of non-Σ operads define strictly associative and A_∞-algebra structures on A

  b. There is a minimal resolution of operads

$$\mathcal{A}_\infty \rightarrow \mathcal{A}ss$$

  c. $\mathcal{A}_\infty$ is realized by $C_*\left(K_n\right)$

- Generalizations:

  a. $\mathcal{A}ss$ extends to the *bialgebra matrad* $\mathcal{H}$

  b. $\mathcal{A}_\infty$ extends to A_∞-*bialgebra matrad* $\mathcal{H}_\infty$

  c. Chain maps

  $$\mathcal{H} \;\to\; Hom\left(H^{\otimes m}, H^{\otimes n}\right)$$

  $$\mathcal{H}_\infty \to Hom\left(H^{\otimes m}, H^{\otimes n}\right)$$

  in the category if matrads define strictly coassociative and A_∞-bialgebra structures on H

  d. There is a minimal resolution of matrads

  $$\mathcal{H}_\infty \to \mathcal{H}$$

  e. $\mathcal{H}_\infty$ is realized by $C_*\left(KK_{n,m}\right)$

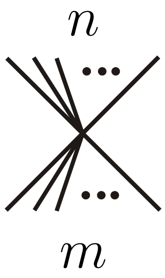
Goal: Specify the combinatorics of $KK_{n,m}$ in dimensions ≤ 3

- $\dim KK_{n,m} = n + m - 3$
- $KK_{n,1} = KK_{1,n} = K_n$ is Stasheff's associahedron

Strategy: Define cellular boundary ∂ on top dim'l faces and extend inductively to lower dimensions.

- $\mathcal{H}_\infty$ has one generator in each bideg (m, n) :

$\theta^n_m =$



thought of as an operation in

$$M = \left\{ M_{n,m} = Hom\left(H^{\otimes m}, H^{\otimes n}\right)\right\}$$

- Data flows upward

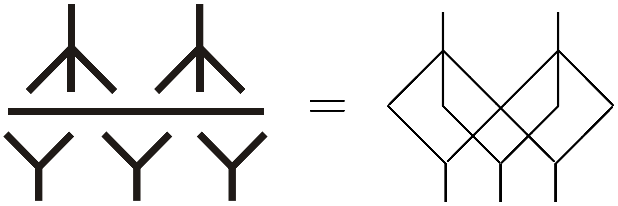
Markl's Fraction Product on TM

Non-zero monomials generated by the θ_m^n's are "fractions" α of the form

$$\alpha = \frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q}$$

in which

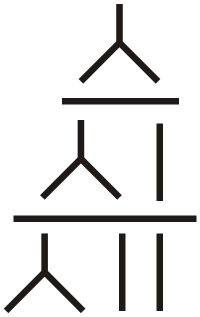
1. There are q factors above $\Leftrightarrow$ each factor below has q outputs
2. There are p factors below $\Leftrightarrow$ each factor above has p inputs
3. The j^{th} output of the i^{th} factor below links to the i^{th} input of j^{th} factor above



- $\dim \alpha = \sum_{i,j} \dim \alpha_{x_i}^q + \dim \alpha_p^{y_j}$

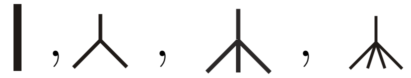
- The fraction product is not associative:

$$\begin{array}{c} \text{A} \\ \hline \text{B} \\ \hline \text{C} \end{array} \quad =$$

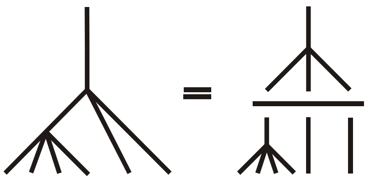


$$(AB)\,C \neq 0 = A\,(BC)$$

- Operad of up-rooted Planar Rooted Trees (PRTs) is the free associative algebra

$A = \Big\langle$  $\dots \Big\rangle \Big/ \sim$

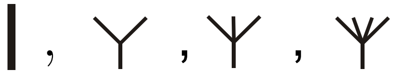
- Monomials in A are classes of fraction products



- Differential on A :

$$\partial\left(\theta_m^1\right) = \sum_{\substack{\alpha_m^1 \in A\cdot A \\ \dim \alpha_m^1 = m-3}} \pm\alpha_m^1$$

- Similarly, the operad of down-rooted PRTs

$C = \Big\langle$  $, \ldots \Big\rangle \Big/ \sim$ with

$$\partial\left(\theta_1^n\right) = \sum_{\substack{\alpha_1^n \in C\cdot C \\ \dim \alpha_n^1 = n-3}} \pm\alpha_1^n$$

- A and C represent the A_∞-operad $\mathcal{A}_\infty$

Matrad Products in dimensions ≤ 3

- Appropriately restrict the fraction product

Given

$$\alpha = \frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q}$$

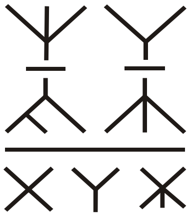
use its *upper and lower leaf sequences*

$$\mathbf{y} = (y_1, \ldots, y_q) \text{ and } \mathbf{x} = (x_1, \ldots, x_p)$$

to decorate α:

$$\alpha_{\mathbf{x}}^{\mathbf{y}} = \alpha$$

Example: $\alpha_{213}^{32} =$



Use leaf sequences to decorate factors:

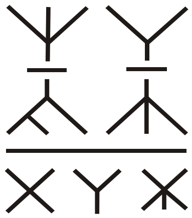
$$\alpha = \frac{\alpha^{\mathbf{y}_1}_{\mathbf{p}_1}\cdots\alpha^{\mathbf{y}_q}_{\mathbf{p}_q}}{\alpha^{\mathbf{q}_1}_{\mathbf{x}_1}\cdots\alpha^{\mathbf{q}_p}_{\mathbf{x}_p}}$$

The *upper and lower contact sequences of* α are

$$(\mathbf{p}_1, \ldots, \mathbf{p}_q) \text{ and } (\mathbf{q}_1, \ldots, \mathbf{q}_p)$$

Example: The *upper and lower contact sequences* of

$\alpha =$

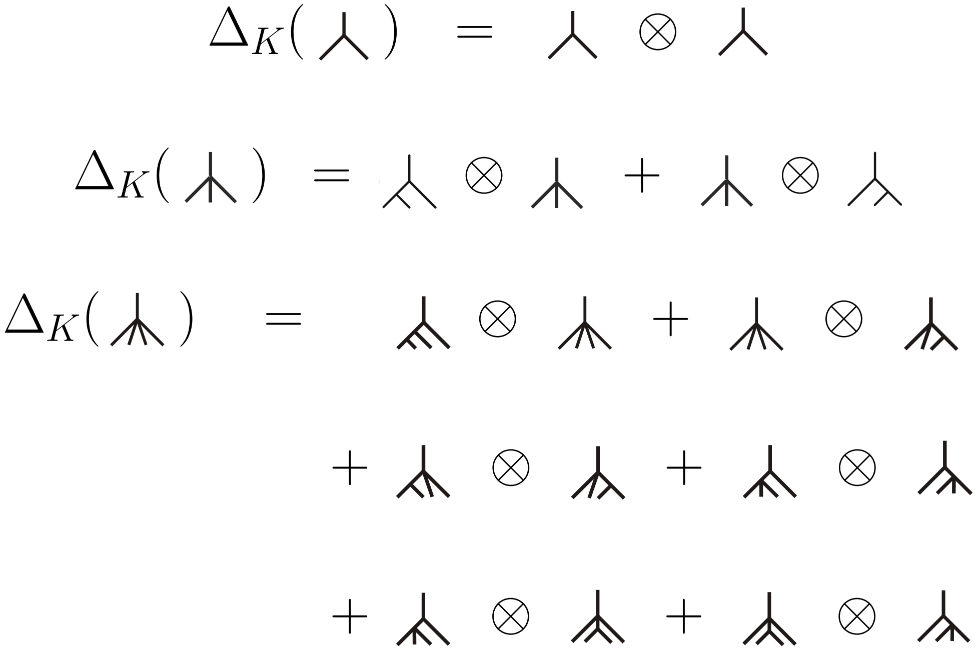


are

$$((2,1),(3)) \text{ and } ((2),(2),(2)).$$

Diagonal on Associahedra (Saneblidze-U)

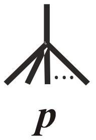
Define $\Delta_K : C_*(K_n) \to C_*(K_n) \otimes C_*(K_n)$ by



Define the *left-iterated diagonal* via

$$\Delta_K^{(0)} = \mathrm{id}$$

$$\Delta_K^{(k)} = \left(\Delta_K \otimes \mathrm{id}^{\otimes k-1}\right) \Delta_K^{(k-1)}$$

- View each component of $\Delta_K^{(q-1)}($ 

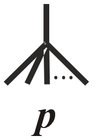
$)$ as

  a $(p-3)$-dim'l subcomplex of $K_p^{\times q}$

- A non-zero *matrad monomial*

$$\alpha = \frac{\alpha_{\mathbf{p}_1}^{\mathbf{y}_1} \cdots \alpha_{\mathbf{p}_q}^{\mathbf{y}_q}}{\alpha_{\mathbf{x}_1}^{\mathbf{q}_1} \cdots \alpha_{\mathbf{x}_p}^{\mathbf{q}_p}}$$

  of dimension ≤ 3 satisfies

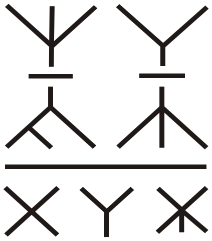
1. UCS $(\mathbf{p}_1, \ldots, \mathbf{p}_q)$ is the list of LLS's in some component of $\Delta_K^{(q-1)}($ 

$)$

2. LCS $(\mathbf{q}_1, \ldots, \mathbf{q}_p)$ is the list of ULS's of some component of $\Delta_K^{(p-1)}($ 

$)$

Example: In

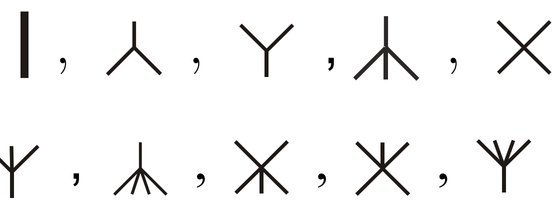
$\alpha \quad =$  ,

UCS $((2,1)\,,(3))$ is the list of LLS's in

 $\otimes$  in $\Delta_K^{(1)}($  $)$

LCS $((2)\,,(2)\,,(2))$ is the list of ULS's in

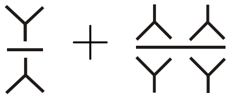
$$\mathsf{Y} \otimes \mathsf{Y} \otimes \mathsf{Y} = \Delta_K^{(2)}(\mathsf{Y})$$

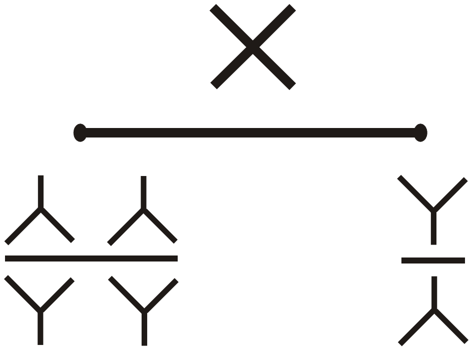
Let $B' = \Big\langle$  $\Big\rangle / \sim$

For $m + n \leq 6$, define

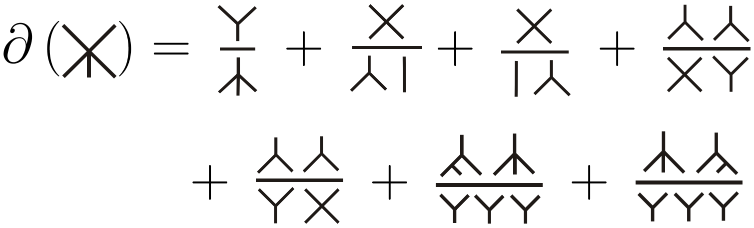
$$\partial\left(\theta_m^n\right) = \sum_{\substack{\alpha_m^n \in B\cdot B \\ \dim \alpha_m^n = m+n-4}} \pm\alpha_m^n$$

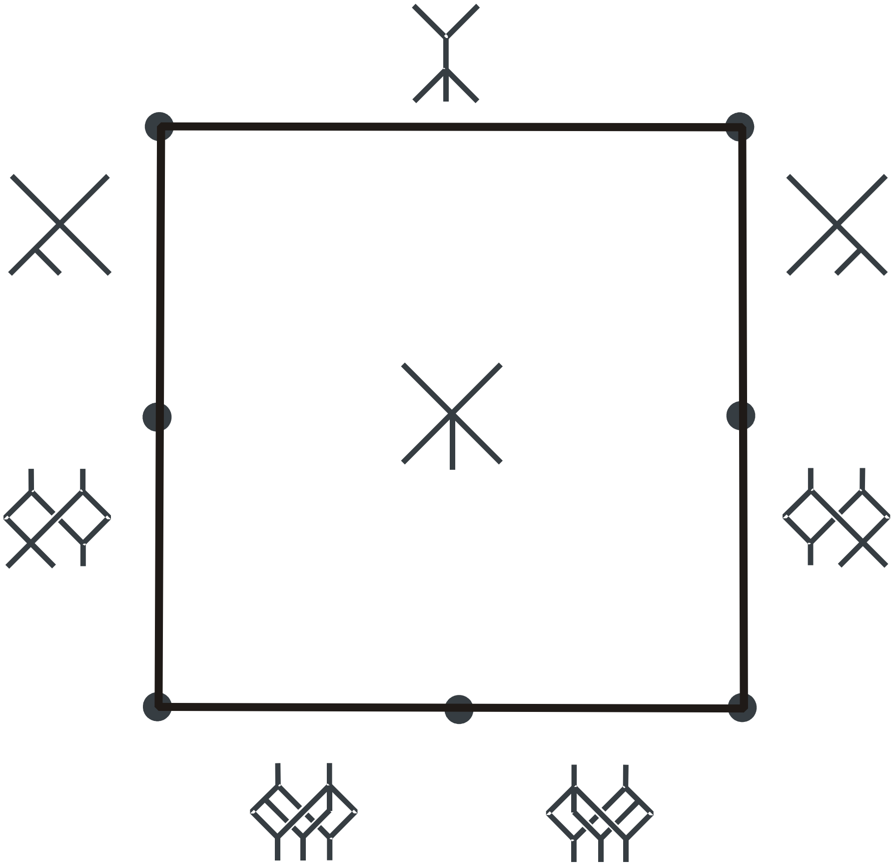
Then

$\partial\,($$) =$ 

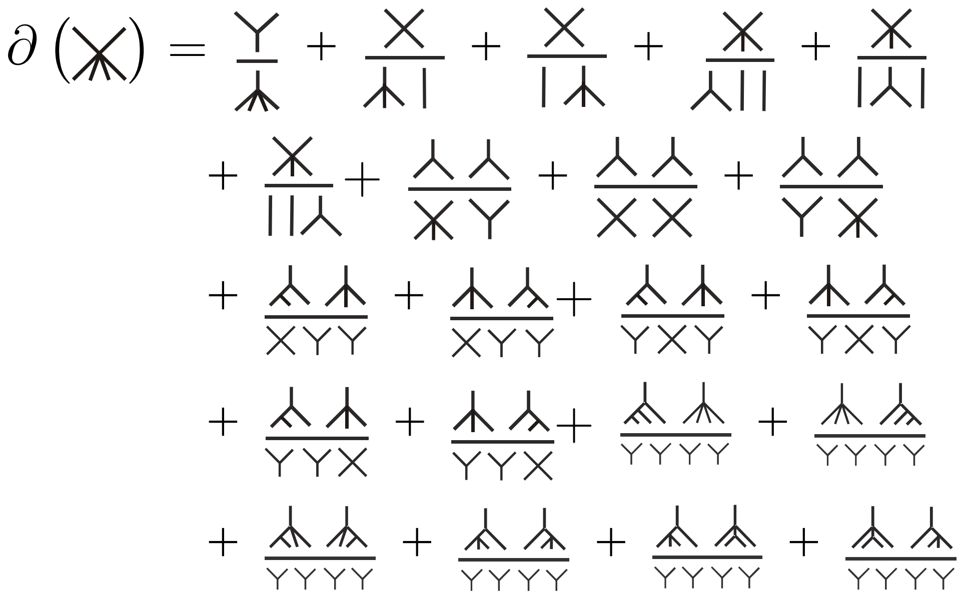


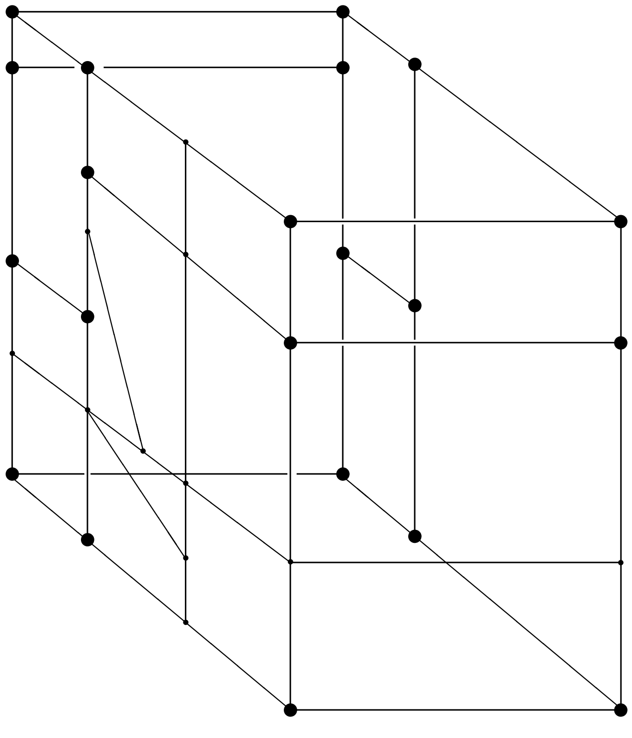
$KK_{2,2}$



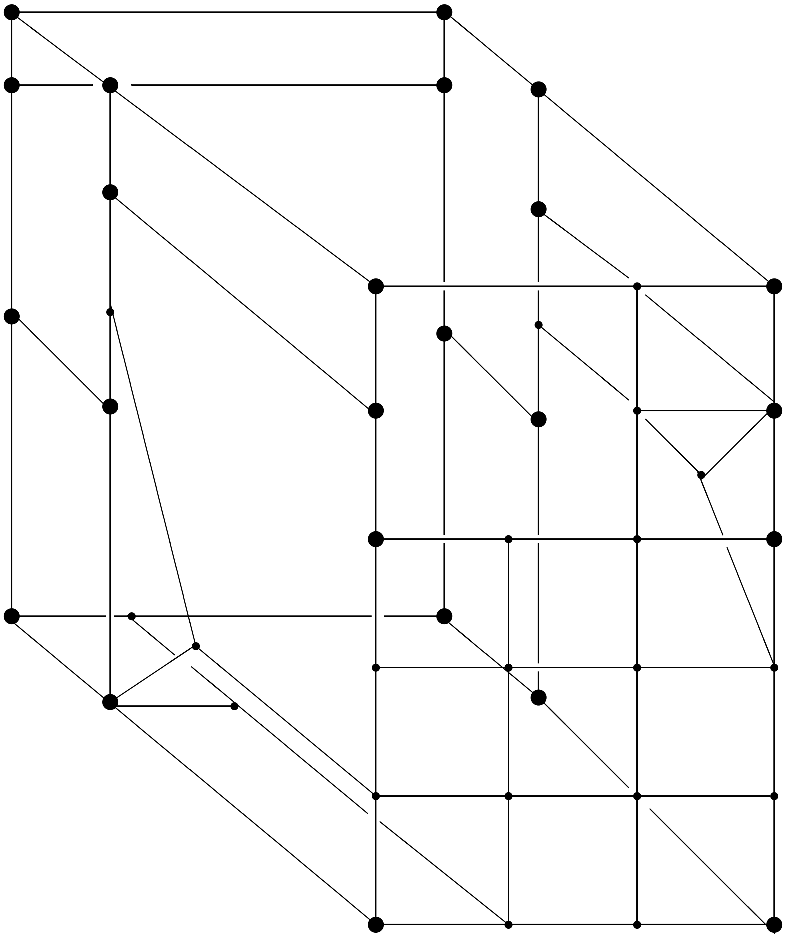


$KK_{2,3}$





$KK_{2,4}$



$KK_{3,3}$

- $C_*(KK)$ realizes the A_∞-*bialgebra matrad* $\mathcal{H}_\infty$

- $End(TH)$ is canonically a matrad

- A map of matrads

$$\mathcal{H}_\infty \to End(TH)$$

defines an A_∞-*bialgebras structure* on H.

- An A_∞-*bialgebra* is an algebra over $\mathcal{H}_\infty$

Alternatively...

The Biderivative

Monomials $\begin{bmatrix} \alpha_{x_1}^{y_1} & \cdots & \alpha_{x_p}^{y_1} \\ \vdots & & \vdots \\ \alpha_{x_1}^{y_q} & \cdots & \alpha_{x_p}^{y_q} \end{bmatrix} \in \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$ are

called *bisequence matrices* and

$$\mathbf{M} = \bigoplus_{\mathbf{x},\mathbf{y}} \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$$

- There is a product $\Upsilon : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{M}$
- For each $m, n \geq 1$, choose

$$\omega_m^n \in Hom\left(H^{\otimes m}, H^{\otimes n}\right)$$

  and let $\omega = \sum \omega_m^n$

- The "biderivative" $d_\omega : M \to \mathbf{M}$ induces a

  non-bilinear operation

  $\circledcirc : M \times M \overset{d_\bullet \times d_\bullet}{\longrightarrow} \mathbf{M} \times \mathbf{M} \overset{\Upsilon}{\longrightarrow} \mathbf{M} \overset{proj}{\to} M$

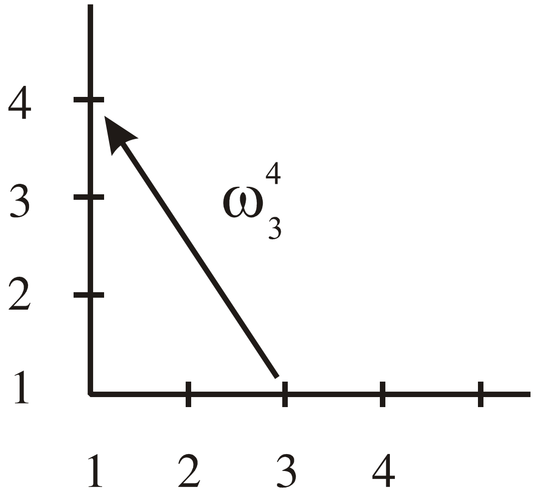
Construct d_ω as follows:

- Linearly extend $d = \omega_1^1$ to $(H^{\otimes p})^{\otimes q}$
- Freely extend the map
  $$\textstyle\sum_{j\geq 1} \omega_1^j : H \to T^a H$$
  as a derivation
- Cofreely extend the map
  $$\textstyle\sum_{i\geq 1} \omega_i^1 : T^c H \to H$$
  as a coderivation
- Freely extend the map
  $$\textstyle\sum_{j>1} \omega_i^j : H^{\otimes i} \to T^a H$$
  as a Δ_P-derivation for each i
- Cofreely extend the map
  $$\textstyle\sum_{i>1} \omega_i^j : T^c H \to H^{\otimes j}$$
  as a Δ_P-coderivation for each j

To picture this, make the identification

$$\left(H^{\otimes p}\right)^{\otimes q} \leftrightarrow (p,q) \in \mathbb{N}^2$$

and represent $\omega_p^q : H^{\otimes p} \to H^{\otimes q}$ as a "transgressive" arrow $(p,1) \to (1,q)$:



- Represent components A and B of the extensions above as arrows in $\mathbb{N}^2$

- When the terminal point of B is the initial point of A define

  $$\gamma\left(A \otimes B\right) = A \circ \sigma_{p,q} \circ B$$

  where $\sigma_{p,q} : \left(H^{\otimes p}\right)^{\otimes q} \overset{\approx}{\to} \left(H^{\otimes q}\right)^{\otimes p}$ is the canonical permutation of tensor factors

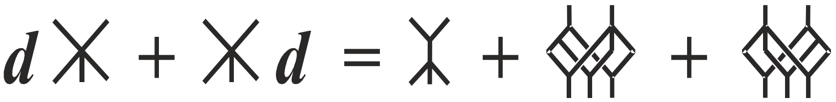
- Define $\omega \circledcirc \omega = \sum_{A,B \in d_\omega} \gamma\left(A \otimes B\right)$

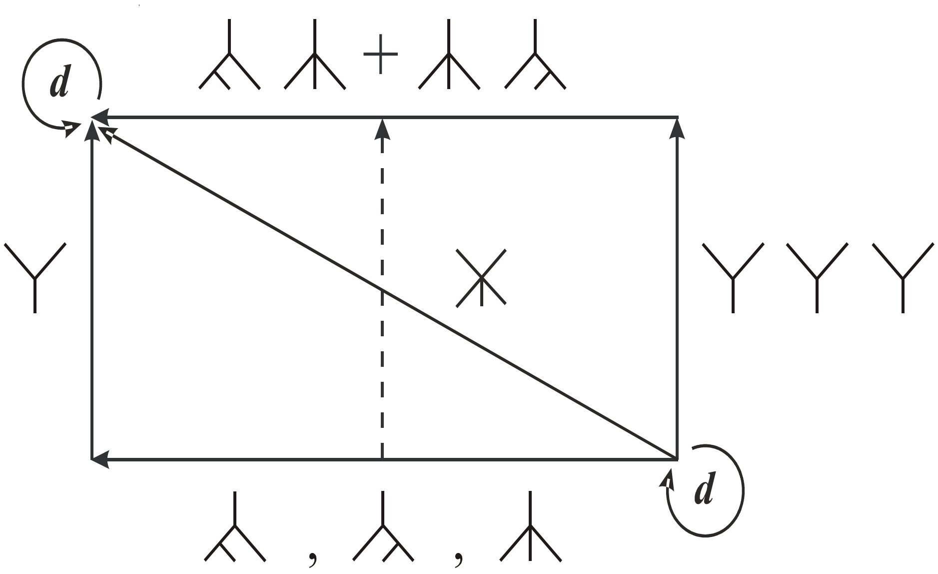
- Summands of $\omega \circledcirc \omega$ have one of two types:
  1. $\gamma(\omega_j^k \otimes \underbrace{(\mathbf{1} \cdots \omega_i^1 \cdots \mathbf{1})}_{j})$ and vise versa
  2. $\gamma\left(A_1 \cdots A_t \otimes B_1 \cdots B_s\right)$ with $s, t \geq 2$

     (sequences of arrows)

- When $\omega \circledcirc \omega = 0$, there is a relation involving certain “transgressive” products $\gamma\,(A \otimes B)$ from $(p,1)$ to $(1,q)$ for each p and q

Example: When $\omega_2^2 = 0$ there is the relation





Alternative Definition:

(H, ω) *is an* A_∞-bialgebra *if* $\omega \circledcirc \omega = 0$

A_∞-Coalgebras and $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$

Given a DG R-module (A, ∂), $\deg \partial = -1$, $\partial^{\otimes n}$ denotes the free linear extension to $A^{\otimes n}$

For each $n \geq 1$, define

$$\delta^n : Hom\left(A, A^{\otimes n}\right) \to Hom\left(A, A^{\otimes n}\right)$$
$$\delta^n(f) = \partial^{\otimes n} f - (-1)^{|f|} f\partial$$

- $(Hom(A, A^{\otimes *}), \delta)$ is a DG R-module

Given $\Delta_2 \in Hom^0\left(A, A^{\otimes 2}\right)$, let
$f = (\Delta_2 \otimes \mathbf{1})\Delta_2$ and $g = (\mathbf{1} \otimes \Delta_2)\Delta_2$

- The cochain $g - f \in Hom^0\left(A, A^{\otimes 3}\right)$ measures the deviation from coassociativity

- A *coassociating homotopy* from f to g is a map $\Delta_3 \in Hom^1\left(A, A^{\otimes 3}\right)$ such that

$$\partial\Delta_3 + \Delta_3\partial^{\otimes 3} = g - f$$

- *A chain map* $\xi : \mathcal{A}_\infty \to Hom\left(A, A^{\otimes *}\right)$ *that preserves operadic structure defines an* A_∞-co*algebra structure on* A.

$\Delta_2 = \xi$ () is a comultiplication

$\Delta_3 = \xi$ () is a coassociating homotopy

$\Delta_4 = \xi$ () is a homotopy of homotopies

Theorem (Berciano) *For all odd primes* p,

$$E(v, 2n+1) \otimes \Gamma(w, 2np+2)$$

is an A_∞*-coalgebra with* $\Delta_q \neq 0$ *iff* $q = 2, p$. *In fact, for* $i = 0, 1$ *and* $\gamma_j = \gamma_j(w)$,

$$\Delta_2(v^i \gamma_j) = \sum_{k=0}^{i} \sum_{l=0}^{j} v^k \gamma_l \otimes v^{i-k} \gamma_{j-l}$$

$$\Delta_p(v^i \gamma_j) = \sum_{k_1 + \cdots + k_p = j-1} v^{i+1} \gamma_{k_1} \otimes \cdots \otimes v^{i+1} \gamma_{k_p}$$

Question 1: Since $E(v, 2n+1) \otimes \Gamma(w, 2np+2)$ is a Hopf algebra, Δ_2 is compatible with the multiplication μ in the sense that Δ_2 is an algebra map. Is Δ_p in some sense compatible with μ?

$\boldsymbol{\Delta}$-Derivations (Saneblidze, U)

Consider an A_∞-coalgebra $(A, \Delta_n)_{n\geq 2}$ with operadic representation ξ and associative multiplication $\mu : A\otimes A \to A$.

- If Δ_2 is an algebra map, i.e.,

$$\begin{aligned}\Delta_2\mu &= \mu^{\otimes 2}\sigma_{2,2}\,(\Delta_2\otimes\Delta_2)\\ &= \mu^{\otimes 2}\sigma_{2,2}\left[(\xi\otimes\xi)\,\Delta_K\left(e^0\right)\right],\end{aligned}$$

  we say that Δ_2 is a Δ-*derivation with respect to the empty family* $\mathfrak{F}_2$ and

$$\mathfrak{F}_3 = \{f = (\Delta_2\otimes 1)\,\Delta_2,\ g = (1\otimes\Delta_2)\,\Delta_2\}$$
$$\leftrightarrow \{\text{faces of } K_3 \text{ with codim } > 0\}$$

  is a Δ-*compatible family on vertices of* K_3.

- If Δ_3 is a (f,g)-derivation, i.e.,

  $$\begin{aligned} \Delta_3\mu &= \mu^{\otimes 3}\sigma_{3,2}\left[f\otimes\Delta_3+\Delta_3\otimes g\right] \\ &= \mu^{\otimes 2}\sigma_{3,2}\left[\left(\xi\otimes\xi\right)\Delta_K\left(e^1\right)\right], \end{aligned}$$

  we say that Δ_3 is a Δ-*derivation with respect to* $\mathfrak{F}_3$ and

  $$\begin{aligned} \mathfrak{F}_4 &= \{\text{compositions involving } \Delta_2, \Delta_3\} \\ &\leftrightarrow \{\text{faces of } K_4 \text{ with codim } > 0\} \end{aligned}$$

  is a Δ-*compatible family on the edges and vertices of* K_4.

Continue inductively:

- If Δ_4 is a Δ-derivation with respect to $\mathfrak{F}_4$,

  i.e., $\Delta_4\mu = \mu^{\otimes 4}\sigma_{4,2}\left[(\xi \otimes \xi)\,\Delta_K\left(e^2\right)\right],$

  $$\mathfrak{F}_5 = \{\text{compositions involving } \Delta_2, \Delta_3, \Delta_4\}$$
  $$\leftrightarrow \{\text{faces of } K_5 \text{ with codim } > 0\}$$

  is a Δ-compatible family on the faces, edges and vertices of K_5.

Definition *A Hopf A_∞-coalgebra is an A_∞-coalgebra $(A, \Delta_n)_{n\geq 2}$ with an associative multiplication μ such thatΔ_n is a Δ-derivation with respect to $\mathfrak{F}_n$ for all n,* i.e.,

$$\Delta_n\mu = \mu^{\otimes n}\sigma_{n,2}\left[(\xi \otimes \xi)\,\Delta_K\left(e^{n-2}\right)\right].$$

Theorem (B-U) *For each* $i \geq 0$, *the* A_∞-*coalgebra* $A_i =$

$$E(v_i, 2n(3^i) + 1) \otimes \Gamma(w_i, 2n(3^{i+1}) + 2)$$

is a Hopf A_∞-*coalgebra with operations* Δ_2, Δ_3 *and* μ.

Proof: With $f = (\Delta_2 \otimes 1)\,\Delta_2 = (1 \otimes \Delta_2)\,\Delta_2$, Δ_3 is an (f, f)-derivation by the Vandermonde Identity.

Theorem (Eilenberg, Mac Lane) For odd primes p,

$H_*(\mathbb{Z}, n; \mathbb{Z}_p) \approx$

$$\bigotimes_{i \geq 0} E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2)$$

Remark: Extending the A_∞-coalgebra on each factor $E \otimes \Gamma$ to a global structure on $H_*(\mathbb{Z}, n; \mathbb{Z}_p)$ requires tensor products of A_∞-coalgebras.

Tensor Products of A_∞-Coalgebras

Definition: (Saneblidze, U) *Given* A_∞-*coalgebras* (A, ξ_A) *and* (B, ξ_B), *define* $\xi_{A\otimes B}$ *as the sum over all* n *of the compositions*

$$\begin{tikzcd}
C_*(K_n) \arrow[r, dashed, "\xi_{A\otimes B}"] \arrow[d, "\Delta_K"'] & Hom(A\otimes B, (A\otimes B)^{\otimes n}) \\
C_*(K_n)\otimes C_*(K_n) \arrow[r, "\xi_A\otimes\xi_B"'] & Hom(A\otimes B, A^{\otimes n}\otimes B^{\otimes n}) \arrow[u, "(\sigma_{n,2})_*"']
\end{tikzcd}$$

The induced operation of order n *is*

$$\Delta_n^* = \xi_{A\otimes B}\left(e^{n-2}\right) : A\otimes B \to (A\otimes B)^{\otimes n},$$

where e^{n-2} *denotes the top dim'l cell of* K_n.

Remark: The tensor product of A_∞-coalgebras is neither cocommutative nor coassociative.

Let $A = E\left(v, 2m+1\right) \otimes \Gamma\left(w, 2mp+2\right)$
and $B = E'\left(v', 2n+1\right) \otimes \Gamma'\left(w', 2np+2\right)$

Theorem (B-U) *The induced operations Δ_2^*, Δ_p^* and Δ_{2p-2}^* act non-trivially on $A \otimes B$.*

Theorem (B-U) *At the prime 3, the induced operations Δ_i^* on $A \otimes B$ vanish for all $i \geq 5$.*

Question 2: Since the tensor product of Hopf algebras is a Hopf algebra, the operation Δ_2^* is compatible with the induced multiplication μ^*. Are Δ_p^* and Δ_{2p-2}^* in some sense compatible with μ^*?

Proposition 4 (B-U) *Let* $(A, \mu_A, \Delta, \Delta_3)$ *and* (B, μ_B, ψ, ψ_3) *be Hopf* A_∞*-coalgebras and let*

$$f = (\Delta \otimes 1)\,\Delta,\ g = (\psi \otimes 1)\,\psi,$$
$$h = \sigma_{3,2}\,(f \otimes g) \text{ and}$$
$$\Delta_3^* = \sigma_{3,2}\,(f \otimes \psi_3 + \Delta_3 \otimes g)\,.$$

Then Δ_3^* *is an* (h, h)*-derivation.*

Proof: Straightforward calculation.

Corollary 5 Let $(A_i, \mu, \Delta, \Delta_3)$ and $(A_{i'}, \mu', \psi, \psi_3)$ be Hopf A_∞-coalgebras of form $E \otimes \Gamma$. Then $(A_i \otimes A_{i'}, \mu^*, \Delta^*, \Delta_3^*)$ is a Hopf A_∞-coalgebra.

Remark: There is an induced operation Δ_4^*.

Corollary 6 $H = H_*\left(\mathbb{Z}, n; \mathbb{Z}_3\right)$ is a Hopf A_∞-coalgebra up to order 3.

Proof: The induced operation Δ_3 on $H \approx \bigotimes_{i\geq 0} A_i$ is the inductive limit of operations $\Delta_3^{(k)}$ on $\left(\bigotimes_{0\leq i\leq k-1} A_i\right) \otimes A_k$.

The Conjecture

Given Hopf A_∞-coalgebras $(A, \mu_A, \Delta, \Delta_3)$ and (B, μ_B, ψ, ψ_3) let

$$\begin{array}{llll} F_1 = & (\Delta \otimes 1 \otimes 1)\,\Delta_3 & G_1 = & (\psi \otimes 1 \otimes 1)\,\psi_3 \\ F_2 = & (1 \otimes \Delta \otimes 1)\,\Delta_3 & G_2 = & (1 \otimes \psi \otimes 1)\,\psi_3 \\ F_3 = & (1 \otimes 1 \otimes \Delta)\,\Delta_3 & G_3 = & (1 \otimes 1 \otimes \psi)\,\psi_3 \\ F_4 = & (\Delta_3 \otimes 1)\,\Delta & G_4 = & (\psi_3 \otimes 1)\,\psi \\ F_5 = & (1 \otimes \Delta_3)\,\Delta & G_5 = & (1 \otimes \psi_3)\,\psi \end{array}$$

Then Stasheff's pentagon relation gives

$$\begin{aligned} 0 &= -F_1 + F_2 - F_3 + F_4 + F_5 \\ &= -G_1 + G_2 - G_3 + G_4 + G_5 \end{aligned}$$

$f = (\Delta \otimes 1 \otimes 1)\,(\Delta \otimes 1)\,\Delta$ and
$g = (\psi \otimes 1 \otimes 1)\,(\psi \otimes 1)\,\psi$ are algebra maps;
F_i is an (f, f)- and G_i is a (g, g)-derivation.
$h = \sigma_{4,2}\,(f \otimes g)$ is an algebra map;
$H_i = \sigma_{4,2}\,(f \otimes G_i + F_i \otimes g)$ is a (h, h)-derivation and $\mathfrak{F}_4 = \{h, H_i\}$ is Δ-compatible.

Let ξ_A and ξ_B be the operadic representation of these A_∞-coalgebra structures. Then

$$\begin{aligned}\Delta_4^* &= \sigma_{4,2}\left(\xi_A \otimes \xi_B\right)\Delta_K\left(e^2\right) \\ &= \sigma_{4,2}(F_2 \otimes G_5 + F_4 \otimes G_2 \\ &\qquad\qquad -F_1 \otimes G_3 + F_4 \otimes G_5).\end{aligned}$$

Conjecture (B-U) Let $A = E\left(v\right) \otimes \Gamma\left(w\right)$ and $B = E\left(v'\right) \otimes \Gamma\left(w'\right)$. Then Δ_4^* is a Δ_K-derivation with respect to $\mathfrak{F}_4$, i.e., the 2-dimensional obstruction Φ_2 arising from the non-primitive terms of $\Delta_K\left(e^2\right)$ vanishes.

Extend this to $H = H_*\left(\mathbb{Z}, n; \mathbb{Z}_3\right)$ inductively. With k tensor factors of form $E \otimes \Gamma$, there is an induced operation Δ_{k+2}^* whose compatibility with μ as a Δ-derivation is measured by a k-dimensional obstruction Φ_k arising from the non-primitive terms of $\Delta_K\left(e^k\right)$. If $\Phi_k = 0$ for all k, then H is a Hopf A_∞-coalgebra.